

# Runtime Analysis of Probabilistic Programs with Unbounded Recursion<sup>\*</sup>

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**Abstract.** We study the runtime in probabilistic programs with unbounded recursion. As underlying formal model for such programs we use *probabilistic pushdown automata (pPDA)* which exactly correspond to recursive Markov chains. We show that every pPDA can be transformed into a stateless pPDA (called “pBPA”) whose runtime and further properties are closely related to those of the original pPDA. This result substantially simplifies the analysis of runtime and other pPDA properties. We prove that for every pPDA the probability of performing a long run decreases *exponentially* in the length of the run, if and only if the expected runtime in the pPDA is *finite*. If the expectation is infinite, then the probability decreases “polynomially”. We show that these bounds are asymptotically tight. Our tail bounds on the runtime are *generic*, i.e., applicable to any probabilistic program with unbounded recursion. An intuitive interpretation is that in pPDA the runtime is exponentially unlikely to deviate from its expected value.

## 1 Introduction

We study the termination time in programs with unbounded recursion, which are either randomized or operate on statistically quantified inputs. As underlying formal model for such programs we use *probabilistic pushdown automata (pPDA)* [15, 16, 7, 4] which are equivalent to recursive Markov chains [20, 18, 19]. Since pushdown automata are a standard and well-established model for programs with recursive procedure calls, our abstract results imply *generic* and *tight* tail bounds for termination time, the main performance characteristic of probabilistic recursive programs.

<sup>\*</sup> This work has been published without proofs as a preliminary version in the *Proceedings of the 38th International Colloquium on Automata, Languages and Programming (ICALP)*, volume 6756 of LNCS, pages 319–331, 2011 at Springer. The presentation has been improved since, and the general lower tail bound has been tightened from  $\Omega(1/n)$  to  $\Omega(1/\sqrt{n})$ .

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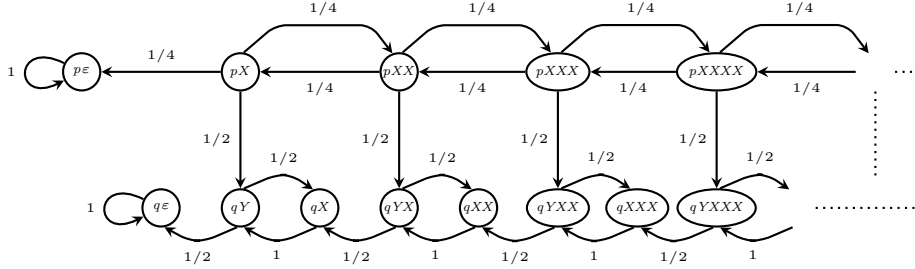
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A pPDA consists of a finite set of *control states*, a finite *stack alphabet*, and a finite set of *rules* of the form  $pX \xrightarrow{x} q\alpha$ , where  $p, q$  are control states,  $X$  is a stack symbol,  $\alpha$  is a finite sequence of stack symbols (possibly empty), and  $x \in (0, 1]$  is the (rational) probability of the rule. We require that for each  $pX$ , the sum of the probabilities of all rules of the form  $pX \xrightarrow{x} q\alpha$  is equal to 1. Each pPDA  $\Delta$  induces an infinite-state Markov chain  $M_\Delta$ , where the states are configurations of the form  $p\alpha$  ( $p$  is the current control state and  $\alpha$  is the current stack content), and  $pX\beta \xrightarrow{x} q\alpha\beta$  is a transition of  $M_\Delta$  iff  $pX \xrightarrow{x} q\alpha$  is a rule of  $\Delta$ . We also stipulate that  $p\varepsilon \xrightarrow{1} p\varepsilon$  for every control state  $p$ , where  $\varepsilon$  denotes the empty stack. For example, consider the pPDA  $\hat{\Delta}$  with two control states  $p, q$ , two stack symbols  $X, Y$ , and the rules

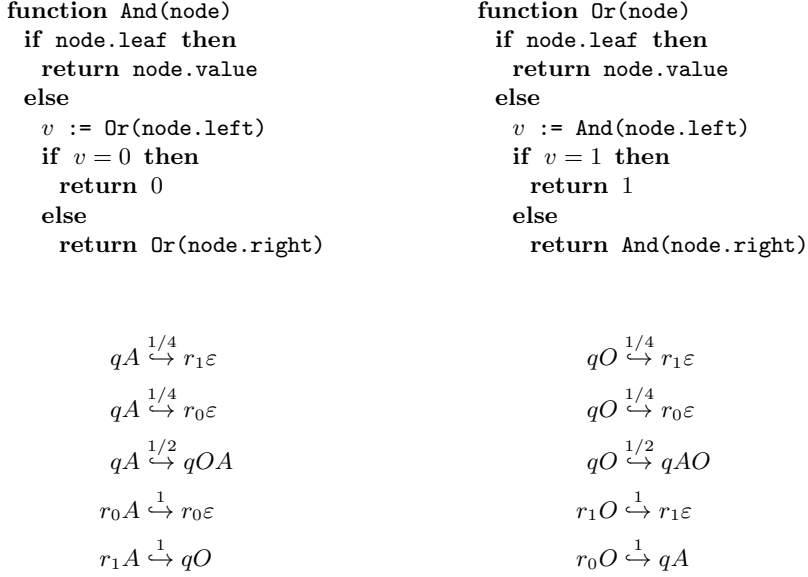
$$pX \xrightarrow{1/4} p\varepsilon, pX \xrightarrow{1/4} pXX, pX \xrightarrow{1/2} qY, pY \xrightarrow{1} pY, qY \xrightarrow{1/2} qX, qY \xrightarrow{1/2} q\varepsilon, qX \xrightarrow{1} qY.$$

The structure of Markov chain  $M_{\hat{\Delta}}$  is indicated below.



pPDA can model programs that use unbounded “stack-like” data structures such as stacks, counters, or even queues (in some cases, the exact ordering of items stored in a queue is irrelevant and the queue can be safely replaced with a stack). Transition probabilities may reflect the random choices of the program (such as “coin flips” in randomized algorithms) or some statistical assumptions about the input data. In particular, pPDA model *recursive* programs. The global data of such a program are stored in the finite control, and the individual procedures and functions together with their local data correspond to the stack symbols (a function call/return is modeled by pushing/popping the associated stack symbol onto/from the stack). As a simple example, consider the recursive program *Tree* of Figure 1, which computes the value of an *And/Or-tree*, i.e., a tree such that (i) every node has either zero or two children, (ii) every inner node is either an And-node or an Or-node, and (iii) on any path from the root to a leaf And- and Or-nodes alternate. We further assume that the root is either a leaf or an And-node. *Tree* starts by invoking the function **And** on the root of a given And/Or-tree. Observe that the program evaluates subtrees only if necessary. Now assume that the input are random And/Or trees following the Galton-Watson distribution: a node of the tree has two children with probability  $1/2$ , and no children with probability  $1/2$ . Furthermore, the conditional probabilities that a childless node evaluates to 0 and 1 are also both equal to  $1/2$ . On inputs with this distribution, the algorithm corresponds to a pPDA  $\Delta_{Tree}$  of Figure 1 (the control states  $r_0$  and  $r_1$  model the return values 0 and 1).

We study the *termination time* of runs in a given pPDA  $\Delta$ . For every pair of control states  $p, q$  and every stack symbol  $X$  of  $\Delta$ , let  $Run(pXq)$  be the set of all runs (infinite paths) in  $M_\Delta$  initiated in  $pX$  which visit  $q\varepsilon$ . The termination time is modeled by the random variable  $\mathbf{T}_{pX}$ , which to every run  $w$  assigns either the number of steps needed



**Fig. 1.** The program *Tree* and its pPDA model  $\Delta_{Tree}$ .

to reach a configuration with empty stack, or  $\infty$  if there is no such configuration. The conditional expected value  $\mathbb{E}[\mathbf{T}_{pX} \mid \text{Run}(pXq)]$ , denoted just by  $E[pXq]$  for short, then corresponds to the average number of steps needed to reach  $q\varepsilon$  from  $pX$ , computed only for those runs initiated in  $pX$  which terminate in  $q\varepsilon$ . For example, using the results of [15, 16, 20], one can show that the functions **And** and **Or** of the program *Tree* terminate with probability one, and the expected termination times can be computed by solving a system of linear equations. Thus, we obtain the following:

$$\begin{array}{lll}
E[qAr_0] = 7.155113 & E[qAr_1] = 7.172218 & \\
E[qOr_0] = 7.172218 & E[qOr_1] = 7.155113 & \\
E[r_0Ar_0] = 1.000000 & E[r_1Ar_0] = 8.172218 & E[r_1Ar_1] = 8.155113 \\
E[r_1Or_1] = 1.000000 & E[r_0Or_1] = 8.172218 & E[r_0Or_0] = 8.155113
\end{array}$$

However, the mere expectation of the termination time does not provide much information about its distribution until we analyze the associated *tail bound*, i.e., the probability that the termination time deviates from its expected value by a given amount. That is, we are interested in bounds for the conditional probability  $\mathcal{P}(\mathbf{T}_{pX} \geq n \mid \text{Run}(pXq))$ . (Note this probability makes sense regardless of whether  $E[pXq]$  is finite or infinite.) Assuming that the (conditional) expectation and variance of  $\mathbf{T}_{pX}$  are finite, one can apply Markov's and Chebyshev's inequalities and thus yield bounds of the form  $\mathcal{P}(\mathbf{T}_{pX} \geq n \mid \text{Run}(pXq)) \leq c/n$  and  $\mathcal{P}(\mathbf{T}_{pX} \geq n \mid \text{Run}(pXq)) \leq c/n^2$ , respectively, where  $c$  is a constant depending only on the underlying pPDA. However, these bounds are asymptotically always worse than our exponential bound (see below). If  $E[pXq]$  is infinite, these inequalities cannot be used at all.

**Our contribution.** The main contributions of this paper are the following:

- We show that every pPDA can be effectively transformed into a *stateless* pPDA (called “pBPA”) so that all important quantitative characteristics of runs are preserved. This simple (but fundamental) observation was overlooked in previous works on pPDA and related models [15, 16, 7, 4, 20, 18, 19], although it simplifies virtually all of these results. Hence, we can w.l.o.g. concentrate just on the study of pBPA. Moreover, for the runtime analysis, the transformation yields a pBPA all of whose symbols terminate with probability one, which further simplifies the analysis.
- We provide tail bounds for  $\mathbf{T}_{pX}$  which are *asymptotically optimal for every pPDA* and are applicable also in the case when  $E[pXq]$  is infinite. More precisely, we show that for every pair of control states  $p, q$  and every stack symbol  $X$ , there are essentially three possibilities:
  - There is a “small”  $k$  such that  $\mathcal{P}(\mathbf{T}_{pX} \geq n \mid \text{Run}(pXq)) = 0$  for all  $n \geq k$ .
  - $E[pXq]$  is finite and  $\mathcal{P}(\mathbf{T}_{pX} \geq n \mid \text{Run}(pXq))$  decreases exponentially in  $n$ .
  - $E[pXq]$  is infinite and  $\mathcal{P}(\mathbf{T}_{pX} \geq n \mid \text{Run}(pXq))$  decreases “polynomially” in  $n$ .

The exact formulation of this result, including the explanation of what is meant by a “polynomial” decrease, is given in Theorem 7 (technically, Theorem 7 is formulated for pBPA which terminate with probability one, which is no restriction as explained above). Observe that a direct consequence of the above theorem is that *all* conditional moments  $\mathbb{E}[\mathbf{T}_{pX}^k \mid \text{Run}(pXq)]$  are simultaneously either finite or infinite (in particular, if  $E[pXq]$  is finite, then so is the conditional variance of  $\mathbf{T}_{pX}$ ).

The characterization given in Theorem 7 is effective. In particular, it is decidable in polynomial space whether  $E[pXq]$  is finite or infinite by using the results of [15, 16, 20], and if  $E[pXq]$  is finite, we can compute concrete bounds on the probabilities. Our results vastly improve on what was previously known on the termination time  $\mathbf{T}_{pX}$ . Previous work, in particular [16, 3], has focused on computing expectations and variances for a class of random variables on pPDA runs, a class that includes  $\mathbf{T}_{pX}$  as prime example. Note that our exponential bound given in Theorem 7 depends, like Markov’s inequality, only on expectations, which can be efficiently approximated by the methods of [16, 14].

An intuitive interpretation of our results is that pPDA with finite (conditional) expected termination time are well-behaved in the sense that the termination time is exponentially unlikely to deviate from its expectation. Of course, a detailed analysis of a concrete pPDA may lead to better bounds, but these bounds will be *asymptotically equivalent* to our generic bounds. Also note that the conditional expected termination time can be finite even for pPDA that do not terminate with probability one. Hence, for every  $\varepsilon > 0$  we can compute a tight threshold  $k$  such that if a given pPDA terminates at all, it terminates after at most  $k$  steps with probability  $1 - \varepsilon$  (this is useful for interrupting programs that are supposed but not guaranteed to terminate).

**Proof techniques.** The main mathematical tool for establishing our results on runtime is (basic) martingale theory and its tools such as the optional stopping theorem and Azuma’s inequality (see Section 4). More precisely, we construct two different martingales corresponding to the cases when the expected termination time is finite resp. infinite. In combination with our reduction to pBPA this establishes a powerful link between pBPA, pPDA, and martingale theory.

Our analysis of termination time in the case when the expected termination time is infinite builds on Perron-Frobenius theory for nonnegative matrices as well as on recent results from [20, 14]. We also use some of the observations presented in [15, 16, 7].

**Related work.** The application of Azuma’s inequality in the analysis of particular randomized algorithms is also known as the *method of bounded differences*; see, e.g., [26, 12] and the references therein. In contrast, we apply martingale methods not to particular algorithms, but to the pPDA model as a whole.

Analyzing the distribution of termination time is closely related to the analysis of multitype branching processes (MT-BPs) [21]. A MT-BP is very much like a pBPA (see above). The stack symbols in pBPA correspond to species in MT-BPs. An  $\varepsilon$ -rule corresponds to the death of an individual, whereas a rule with two or more symbols on the right hand side corresponds to reproduction. Since in MT-BPs the symbols on the right hand side of rules evolve concurrently, termination time in pBPA does *not* correspond to extinction time in MT-BPs, but to the size of the *total progeny* of an individual, i.e., the number of direct or indirect descendants of an individual. The distribution of the total progeny of a MT-BP has been studied mainly for the case of a single species, see, e.g., [21, 27, 28] and the references therein, but to the best of our knowledge, no tail bounds for MT-BPs have been given. Hence, Theorem 7 can also be seen as a contribution to MT-BP theory.

Stochastic context-free grammars (SCFGs) [25] are also closely related to pBPA. The termination time in pBPA corresponds to the number of nodes in a derivation tree of a SCFG, so our analysis of pBPA immediately applies to SCFGs. Quasi-Birth-Death processes (QBDs) can also be seen as a special case of pPDA. A QBD is a generalization of a birth-death process studied in queueing theory and applied probability (see, e.g., [24, 2, 17]). Intuitively, a QBD describes an unbounded queue, using a counter to count the number of jobs in the queue, where the queue can be in one of finitely many distinct “modes”. Hence, a (discrete-time) QBD can be equivalently defined by a pPDA with one stack symbol used to emulate the counter. These special pPDA are also known as *probabilistic one-counter automata (pOC)* [17, 6, 5]. Recently, it has been shown in [8] that every pOC induces a martingale apt for studying the properties of both terminating and nonterminating runs in pOC. The construction is based on ideas specific to pOC that are completely unrelated to the ones presented in this paper.

Previous work on pPDA and the equivalent model of recursive Markov chains includes [15, 16, 7, 4, 20, 18, 19]. In this paper we use many of the results presented in these papers, which is explicitly acknowledged at appropriate places.

**Organization of the paper.** We present our results after some preliminaries in Section 2. In Section 3 we show how to transform a given pPDA into an equivalent pBPA, and in Section 4 we design the promised martingales and derive tight tail bounds for the termination time. We conclude in Section 5. Some proofs have been moved to Section 6.

## 2 Preliminaries

In the rest of this paper,  $\mathbb{N}$ ,  $\mathbb{N}_0$ , and  $\mathbb{R}$  denote the set of positive integers, non-negative integers, and real numbers, respectively. The tuples of  $A_1 \times A_2 \cdots \times A_n$  are often written

simply as  $a_1a_2 \dots a_n$ . The set of all finite words over a given alphabet  $\Sigma$  is denoted by  $\Sigma^*$ , and the set of all infinite words over  $\Sigma$  is denoted by  $\Sigma^\omega$ . We write  $\varepsilon$  for the empty word. The length of a given  $w \in \Sigma^* \cup \Sigma^\omega$  is denoted by  $|w|$ , where the length of an infinite word is  $\infty$ . Given a word (finite or infinite) over  $\Sigma$ , the individual letters of  $w$  are denoted by  $w(0), w(1), \dots$ . For  $X \in \Sigma$  and  $w \in \Sigma^*$ , we denote by  $\#(X)(w)$  the number of occurrences of  $X$  in  $w$ .

**Definition 1 (Markov Chains).** A Markov chain is a triple  $M = (S, \rightarrow, Prob)$  where  $S$  is a finite or countably infinite set of states,  $\rightarrow \subseteq S \times S$  is a transition relation, and  $Prob$  is a function which to each transition  $s \rightarrow t$  of  $M$  assigns its probability  $Prob(s \rightarrow t) > 0$  so that for every  $s \in S$  we have  $\sum_{s \rightarrow t} Prob(s \rightarrow t) = 1$  (as usual, we write  $s \xrightarrow{x} t$  instead of  $Prob(s \rightarrow t) = x$ ).

A path in  $M$  is a finite or infinite word  $w \in S^+ \cup S^\omega$  such that  $w(i-1) \rightarrow w(i)$  for every  $1 \leq i < |w|$ . For a state  $s$ , we use  $FPath(s)$  to denote the set of all finite paths initiated in  $s$ . A run in  $M$  is an infinite path in  $M$ . We denote by  $Run[M]$  the set of all runs in  $M$ . The set of all runs that start with a given finite path  $w$  is denoted by  $Run[M](w)$ . When  $M$  is understood, we write just  $Run$  and  $Run(w)$  instead of  $Run[M]$  and  $Run[M](w)$ , respectively. Given  $s \in S$  and  $A \subseteq S$ , we say  $A$  is reachable from  $s$  if there is a run  $w$  such that  $w(0) = s$  and  $w(i) \in A$  for some  $i \geq 0$ .

To every  $s \in S$  we associate the probability space  $(Run(s), \mathcal{F}, \mathcal{P})$  where  $\mathcal{F}$  is the  $\sigma$ -field generated by all basic cylinders  $Run(w)$  where  $w$  is a finite path starting with  $s$ , and  $\mathcal{P} : \mathcal{F} \rightarrow [0, 1]$  is the unique probability measure such that  $\mathcal{P}(Run(w)) = \prod_{i=1}^{|w|-1} x_i$  where  $w(i-1) \xrightarrow{x_i} w(i)$  for every  $1 \leq i < |w|$ . If  $|w| = 1$ , we put  $\mathcal{P}(Run(w)) = 1$ . Note that only certain subsets of  $Run(s)$  are  $\mathcal{P}$ -measurable, but in this paper we only deal with “safe” subsets that are guaranteed to be in  $\mathcal{F}$ .

**Definition 2 (probabilistic PDA).** A probabilistic pushdown automaton (pPDA) is a tuple  $\Delta = (Q, \Gamma, \hookrightarrow, Prob)$  where  $Q$  is a finite set of control states,  $\Gamma$  is a finite stack alphabet,  $\hookrightarrow \subseteq (Q \times \Gamma) \times (Q \times \Gamma^{\leq 2})$  is a transition relation (where  $\Gamma^{\leq 2} = \{\alpha \in \Gamma^*, |\alpha| \leq 2\}$ ), and  $Prob$  is a function which to each transition  $pX \hookrightarrow q\alpha$  assigns its probability  $Prob(pX \hookrightarrow q\alpha) > 0$  so that for all  $p \in Q$  and  $X \in \Gamma$  we have that  $\sum_{pX \hookrightarrow q\alpha} Prob(pX \hookrightarrow q\alpha) = 1$ . As usual, we write  $pX \xrightarrow{x} q\alpha$  instead of  $Prob(pX \hookrightarrow q\alpha) = x$ .

Elements of  $Q \times \Gamma^*$  are called configurations of  $\Delta$ . A pPDA with just one control state is called pBPA.<sup>6</sup> In what follows, configurations of pBPA are usually written without the (only) control state  $p$  (i.e., we write just  $\alpha$  instead of  $p\alpha$ ). We define the size of a pPDA  $\Delta$  as  $|\Delta| = |Q| + |\Gamma| + |\hookrightarrow| + |Prob|$ , where  $|Prob|$  is the sum of sizes of binary representations of values taken by  $Prob$ . To  $\Delta$  we associate the Markov chain  $M_\Delta$  with  $Q \times \Gamma^*$  as the set of states and transitions defined as follows:

- $p\varepsilon \xrightarrow{1} p\varepsilon$  for each  $p \in Q$ ;
- $pX\beta \xrightarrow{x} q\alpha\beta$  is a transition of  $M_\Delta$  iff  $pX \xrightarrow{x} q\alpha$  is a transition of  $\Delta$ .

For all  $pXq \in Q \times \Gamma \times Q$  and  $rY \in Q \times \Gamma$ , we define

<sup>6</sup> The “BPA” acronym stands for “Basic Process Algebra” and it is used mainly for historical reasons. pBPA are closely related to stochastic context-free grammars and are also called *1-exit recursive Markov chains* (see, e.g., [20]).

- $Run(pXq) = \{w \in Run(pX) \mid w(i) = q\varepsilon \text{ for some } i \in \mathbb{N}\}$
- $Run(rY\uparrow) = Run(rY) \setminus \bigcup_{s \in Q} Run(rYs)$ .

Further, we put  $[pXq] = \mathcal{P}(Run(pXq))$  and  $[pX\uparrow] = \mathcal{P}(Run(pX\uparrow))$ . If  $\Delta$  is a pBPA, we write  $[X]$  and  $[X\uparrow]$  instead of  $[pXp]$  and  $[pX\uparrow]$ , where  $p$  is the only control state of  $\Delta$ .

Let  $p\alpha \in Q \times \Gamma^*$ . We denote by  $\mathbf{T}_{p\alpha}$  a random variable over  $Run(p\alpha)$  where  $\mathbf{T}_{p\alpha}(w)$  is either the least  $n \in \mathbb{N}_0$  such that  $w(n) = q\varepsilon$  for some  $q \in Q$ , or  $\infty$  if there is no such  $n$ . Intuitively,  $\mathbf{T}_{p\alpha}(w)$  is the number of steps (“the time”) in which the run  $w$  initiated in  $p\alpha$  terminates. We write  $E[p\alpha] := \mathbb{E}[\mathbf{T}_{p\alpha}]$  for the expected termination time (usually omitting the control state  $p$  for pBPA).

### 3 Transforming pPDA into pBPA

Let  $\Delta = (Q, \Gamma, \hookrightarrow, Prob)$  be a pPDA. We show how to construct a pBPA  $\Delta_\bullet$  which is “equivalent” to  $\Delta$  in a well-defined sense. This construction is a relatively straightforward modification of the standard method for transforming a PDA into an equivalent context-free grammar (see, e.g., [22]), but has so far been overlooked in the existing literature on probabilistic PDA. The idea behind this method is to construct a BPA with stack symbols of the form  $\langle pXq \rangle$  for all  $p, q \in Q$  and  $X \in \Gamma$ . Roughly speaking, such a triple corresponds to terminating paths from  $pX$  to  $q\varepsilon$ . Subsequently, transitions of the BPA are induced by transitions of the PDA in a way corresponding to this intuition. For example, a transition of the form  $pX \hookrightarrow rYZ$  induces transitions of the form  $\langle pXq \rangle \hookrightarrow \langle rYs \rangle \langle sZq \rangle$  for all  $s \in Q$ . Then each path from  $pX$  to  $q\varepsilon$  maps naturally to a path from  $\langle pXq \rangle$  to  $\varepsilon$ . This construction can also be applied in the probabilistic setting by assigning probabilities to transitions so that the probability of the corresponding paths is preserved. We also deal with nonterminating runs by introducing new stack symbols of the form  $\langle pX\uparrow \rangle$ .

Formally, the stack alphabet of  $\Delta_\bullet$  is defined as follows: For every  $pX \in Q \times \Gamma$  such that  $[pX\uparrow] > 0$  we add a stack symbol  $\langle pX\uparrow \rangle$ , and for every  $pXq \in Q \times \Gamma \times Q$  such that  $[pXq] > 0$  we add a stack symbol  $\langle pXq \rangle$ . Note that the stack alphabet of  $\Delta_\bullet$  is effectively constructible in polynomial space by applying the results of [15, 20].

Now we construct the rules  $\hookrightarrow_\bullet$  of  $\Delta_\bullet$ . For all  $\langle pXq \rangle$  we have the following rules:

- if  $pX \xrightarrow{x} rYZ$  in  $\Delta$ , then for all  $s \in Q$  such that  $y = x \cdot [rYs] \cdot [sZq] > 0$  we put  $\langle pXq \rangle \xrightarrow{y/[pXq]}_\bullet \langle rYs \rangle \langle sZq \rangle$ ;
- if  $pX \xrightarrow{x} rY$  in  $\Delta$ , where  $y = x \cdot [rYq] > 0$ , we put  $\langle pXq \rangle \xrightarrow{y/[pXq]}_\bullet \langle rYq \rangle$ ;
- if  $pX \xrightarrow{x} q\varepsilon$  in  $\Delta$ , we put  $\langle pXq \rangle \xrightarrow{x/[pXq]}_\bullet \varepsilon$ .

For all  $\langle pX\uparrow \rangle$  we have the following rules:

- if  $pX \xrightarrow{x} rYZ$  in  $\Delta$ , then for every  $s \in Q$  where  $y = x \cdot [rYs] \cdot [sZ\uparrow] > 0$  we add  $\langle pX\uparrow \rangle \xrightarrow{y/[pX\uparrow]}_\bullet \langle rYs \rangle \langle sZ\uparrow \rangle$ ;
- for all  $qY \in Q \times \Gamma$  where  $x = [qY\uparrow] \cdot \sum_{pX \hookrightarrow qY\beta} Prob(pX \hookrightarrow qY\beta) > 0$ , we add  $\langle pX\uparrow \rangle \xrightarrow{x/[pX\uparrow]}_\bullet \langle qY\uparrow \rangle$ .

Note that the transition probabilities of  $\Delta_\bullet$  may take irrational values. Still, the construction of  $\Delta_\bullet$  is to some extent “effective” due to the following proposition:

**Proposition 3 ([15, 20]).** *Let  $\Delta = (Q, \Gamma, \hookrightarrow, Prob)$  be a pPDA. Let  $pXq \in Q \times \Gamma \times Q$ . There is a formula  $\Phi(x)$  of  $ExTh(\mathbb{R})$  (the existential theory of the reals) with one free variable  $x$  such that the length of  $\Phi(x)$  is polynomial in  $|\Delta|$  and  $\Phi(x/r)$  is valid iff  $r = [pXq]$ .*

Using Proposition 3, one can compute formulae of  $ExTh(\mathbb{R})$  that “encode” transition probabilities of  $\Delta_\bullet$ . Moreover, these probabilities can be effectively approximated up to an arbitrarily small error by employing either the decision procedure for  $ExTh(\mathbb{R})$  [10] or by using Newton’s method [13, 23, 14].

*Example 4.* Consider a pPDA  $\Delta$  with two control states,  $p, q$ , one stack symbol,  $X$ , and the following transition rules:

$$pX \xrightarrow{a} qXX, pX \xrightarrow{1-a} q\varepsilon, qX \xrightarrow{b} pXX, qX \xrightarrow{1-b} p\varepsilon,$$

where both  $a, b$  are greater than  $1/2$ . Apparently,  $[pXp] = [qXq] = 0$ . Using results of [15] one can easily verify that  $[pXq] = (1-a)/b$  and  $[qXp] = (1-b)/a$ . Thus  $[pX\uparrow] = (a+b-1)/b$  and  $[qX\uparrow] = (a+b-1)/a$ . Thus the stack symbols of  $\Delta_\bullet$  are  $\langle pXq \rangle, \langle qXp \rangle, \langle pX\uparrow \rangle, \langle qX\uparrow \rangle$ . The transition rules of  $\Delta_\bullet$  are:

$$\begin{array}{llll} \langle pXq \rangle \xrightarrow{1-b}_\bullet \langle qXp \rangle \langle pXq \rangle & \langle pXq \rangle \xrightarrow{b}_\bullet \varepsilon & \langle qXp \rangle \xrightarrow{1-a}_\bullet \langle pXq \rangle \langle qXp \rangle & \langle qXp \rangle \xrightarrow{a}_\bullet \varepsilon \\ \langle pX\uparrow \rangle \xrightarrow{1-b}_\bullet \langle qXp \rangle \langle pX\uparrow \rangle & \langle pX\uparrow \rangle \xrightarrow{b}_\bullet \langle qX\uparrow \rangle & \langle qX\uparrow \rangle \xrightarrow{1-a}_\bullet \langle pXq \rangle \langle qX\uparrow \rangle & \langle qX\uparrow \rangle \xrightarrow{a}_\bullet \langle pX\uparrow \rangle \end{array}$$

As both  $a, b$  are greater than  $1/2$ , the resulting pBPA has a tendency to remove symbols rather than add symbols. Thus both  $\langle pXq \rangle$  and  $\langle qXp \rangle$  terminate with probability 1.

When studying long-run properties of pPDA (such as  $\omega$ -regular properties or limit-average properties), one usually assumes that the runs are initiated in a configuration  $p_0X_0$  which cannot terminate, i.e.,  $[p_0X_0\uparrow] = 1$ . Under this assumption, the probability spaces over  $Run[M_\Delta](p_0X_0)$  and  $Run[M_{\Delta_\bullet}](\langle p_0X_0\uparrow \rangle)$  are “isomorphic” w.r.t. all properties that depend only on the control states and the top-of-the-stack symbols of the configurations visited along a run. This is formalized in our next proposition.

**Proposition 5.** *Let  $p_0X_0 \in Q \times \Gamma$  such that  $[p_0X_0\uparrow] = 1$ . Then there is a partial function  $\Upsilon : Run[M_\Delta](p_0X_0) \rightarrow Run[M_{\Delta_\bullet}](\langle p_0X_0\uparrow \rangle)$  such that for every  $w \in Run[M_\Delta](p_0X_0)$ , where  $\Upsilon(w)$  is defined, and every  $n \in \mathbb{N}$  we have the following: if  $w(n) = qY\beta$ , then  $\Upsilon(w)(n) = \langle qY\uparrow \rangle \gamma$ , where  $\uparrow$  is either an element of  $Q$  or  $\uparrow$ . Further, for every measurable set of runs  $R \subseteq Run[M_{\Delta_\bullet}](\langle p_0X_0\uparrow \rangle)$  we have that  $\Upsilon^{-1}(R)$  is measurable and  $\mathcal{P}(R) = \mathcal{P}(\Upsilon^{-1}(R))$ .*

As for terminating runs, observe that the “terminating” symbols of the form  $\langle pXq \rangle$  do not depend on the “nonterminating” symbols of the form  $\langle pX\uparrow \rangle$ , i.e., if we restrict  $\Delta_\bullet$  just to terminating symbols, we again obtain a pBPA. A straightforward computation reveals the following proposition about terminating runs that is crucial for our results presented in the next section.

**Proposition 6.** *Let  $pXq \in Q \times \Gamma \times Q$  and  $[pXq] > 0$ . Then almost all runs of  $M_{\Delta_\bullet}$  initiated in  $\langle pXq \rangle$  terminate, i.e., reach  $\varepsilon$ . Further, for all  $n \in \mathbb{N}$  we have that*

$$\mathcal{P}(\mathbf{T}_{pX} = n \mid Run(pXq)) = \mathcal{P}(\mathbf{T}_{\langle pXq \rangle} = n \mid Run(\langle pXq \rangle))$$



Observe that this proposition, together with a very special form of rules in  $\Delta_\bullet$ , implies that all configurations reachable from a nonterminating configuration  $p_0 X_0$  have the form  $\alpha \langle qY \uparrow \rangle$ , where  $\alpha$  terminates almost surely and  $\langle qY \uparrow \rangle$  never terminates. It follows that such a pBPA can be transformed into a finite-state Markov chain (whose states are the nonterminating symbols) which is allowed to make recursive calls that almost surely terminate (using rules of the form  $\langle pX \uparrow \rangle \hookrightarrow \langle rZq \rangle \langle qY \uparrow \rangle$ ). This observation is very useful when investigating the properties of nonterminating runs, and many of the existing results about pPDA can be substantially simplified using this result.

## 4 Analysis of pBPA

In this section we establish the promised tight tail bounds for the termination time. By virtue of Proposition 6, it suffices to analyze *almost surely terminating* pBPA, i.e., pBPA all whose stack symbols terminate with probability 1. In what follows we assume that  $\Delta$  is such a pBPA, and we also fix an initial stack symbol  $X_0$ . For  $X, Y \in \Gamma$ , we say that  $X$  *depends directly on*  $Y$ , if there is a rule  $X \hookrightarrow \alpha$  such that  $Y$  occurs in  $\alpha$ . Further, we say that  $X$  *depends on*  $Y$ , if either  $X$  depends directly on  $Y$ , or  $X$  depends directly on a symbol  $Z \in \Gamma$  which depends on  $Y$ . One can compute, in linear time, the directed acyclic graph (DAG) of strongly connected components (SCCs) of the dependence relation. The *height* of this DAG, denoted by  $h$ , is defined as the longest distance between a top SCC and a bottom SCC plus 1 (i.e.,  $h = 1$  if there is only one SCC). We can safely assume that all symbols on which  $X_0$  does not depend were removed from  $\Delta$ . We abbreviate  $\mathcal{P}(\mathbf{T}_{X_0} \geq n \mid \text{Run}(X_0))$  to  $\mathcal{P}(\mathbf{T}_{X_0} \geq n)$ , and we use  $p_{\min}$  to denote  $\min\{p \mid X \xrightarrow{p} \alpha \text{ in } \Delta\}$ . Here is our main result:

**Theorem 7.** *Let  $\Delta$  be an almost surely terminating pBPA with stack alphabet  $\Gamma$ . Assume that  $X_0 \in \Gamma$  depends on all  $X \in \Gamma \setminus \{X_0\}$ , and let  $p_{\min} = \min\{p \mid X \xrightarrow{p} \alpha \text{ in } \Delta\}$ . Then one of the following is true:*

- (1)  $\mathcal{P}(\mathbf{T}_{X_0} \geq 2^{|\Gamma|}) = 0$ .
- (2)  $E[X_0]$  is **finite** and for all  $n \in \mathbb{N}$  with  $n \geq 2E[X_0]$  we have that

$$p_{\min}^n \leq \mathcal{P}(\mathbf{T}_{X_0} \geq n) \leq \exp\left(1 - \frac{n}{8E_{\max}^2}\right)$$

where  $E_{\max} = \max_{X \in \Gamma} E[X]$ .

- (3)  $E[X_0]$  is **infinite** and there is  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$  we have that

$$c/n^{1/2} \leq \mathcal{P}(\mathbf{T}_{X_0} \geq n) \leq d_1/n^{d_2}$$

where  $d_1 = 18h|\Gamma|/p_{\min}^{3|\Gamma|}$ , and  $d_2 = 1/(2^{h+1} - 2)$ . Here,  $h$  is the height of the DAG of SCCs of the dependence relation, and  $c$  is a suitable positive constant depending on  $\Delta$ .

More colloquially, Theorem 7 states that  $\Delta$  satisfies either (1) or (2) or (3), where (1) is when  $\Delta$  does not have any long terminating runs; and (2) resp. (3) is when the expected termination time is finite (resp. infinite) and the probability of performing a terminating run of length  $n$  decreases exponentially (resp. polynomially) in  $n$ .

One can effectively distinguish between the three cases set out in Theorem 7. More precisely, case (1) can be recognized in polynomial time by looking only at the structure

of the pBPA, i.e., disregarding the probabilities. Determining whether  $E[X_0]$  is finite or infinite can be done in polynomial space by employing the results of [16, 3]. This holds even if the transition probabilities of  $\Delta$  are represented just symbolically by formulae of  $ExTh(\mathbb{R})$  (see Proposition 3).

The proof of Theorem 7 is based on designing suitable martingales that are used to analyze the concentration of the termination time. Recall that a *martingale* is an infinite sequence of random variables  $m^{(0)}, m^{(1)}, \dots$  such that, for all  $i \in \mathbb{N}$ ,  $\mathbb{E}[|m^{(i)}|] < \infty$ , and  $\mathbb{E}[m^{(i+1)} \mid m^{(1)}, \dots, m^{(i)}] = m^{(i)}$  almost surely. If  $|m^{(i)} - m^{(i-1)}| < c_i$  for all  $i \in \mathbb{N}$ , then we have the following *Azuma's inequality* (see, e.g., [29]):

$$\mathcal{P}(m^{(n)} - m^{(0)} \geq t) \leq \exp\left(\frac{-t^2}{2 \sum_{k=1}^n c_k^2}\right)$$

We split the proof of Theorem 7 into four propositions (namely Propositions 8–11 below), which together imply Theorem 7.

The following proposition establishes the lower bound from Theorem 7 (2):

**Proposition 8.** *Let  $\Delta$  be an almost surely terminating pBPA with stack alphabet  $\Gamma$ . Let  $p_{\min} = \min\{p \mid X \xrightarrow{p} \alpha \text{ in } \Delta\}$ . Assume that  $\mathcal{P}(\mathbf{T}_{X_0} \geq 2^{|\Gamma|}) > 0$ . Then we have*

$$p_{\min}^n \leq \mathcal{P}(\mathbf{T}_{X_0} \geq n) \quad \text{for all } n \in \mathbb{N}.$$

*Proof.* Let  $\mathbf{T}_{X_0}(w) \geq n$  for some  $n \in \mathbb{N}$  and some  $w \in \text{Run}(X_0)$ . It follows from the definition of the probability space of a pPDA that the set of all runs starting with  $w(0), w(1), \dots, w(n)$  has a probability of at least  $p_{\min}^n$ . Therefore, in order to complete the proof, it suffices to show that  $\mathcal{P}(\mathbf{T}_{X_0} \geq 2^{|\Gamma|}) > 0$  implies  $\mathcal{P}(\mathbf{T}_{X_0} \geq n) > 0$  for all  $n \in \mathbb{N}$ .

To this end, we use a form of the pumping lemma for context-free languages. Notice that a pBPA can be regarded as a context-free grammar with probabilities (a stochastic context-free grammar) with an empty set of terminal symbols and  $\Gamma$  as the set of nonterminal symbols. Each finite run  $w \in \text{Run}(X_0)$  corresponds to a derivation tree with root  $X_0$  that derives the word  $\varepsilon$ . The termination time  $\mathbf{T}_{X_0}$  is the number of (internal) nodes in the tree. In the rest of the proof we use this correspondence.

Let  $\mathcal{P}(\mathbf{T}_{X_0} \geq 2^{|\Gamma|}) > 0$ . Then there is a run  $w \in \text{Run}(X_0)$  with  $\mathbf{T}_{X_0}(w) \geq 2^{|\Gamma|}$ . This run  $w$  corresponds to a derivation tree with at least  $2^{|\Gamma|}$  (internal) nodes. In this tree there is a path from the root (labeled with  $X_0$ ) to a leaf such that on this path there are two different nodes, both labeled with the same symbol. Let us call those nodes  $n_1$  and  $n_2$ , where  $n_1$  is the node closer to the root. By replacing the subtree rooted at  $n_2$  with the subtree rooted at  $n_1$  we obtain a larger derivation tree. This completes the proof.  $\square$

The following proposition establishes the upper bound of Theorem 7 (2):

**Proposition 9.** *Let  $\Delta$  be an almost surely terminating pBPA with stack alphabet  $\Gamma$ . Assume that  $X_0$  depends on all  $X \in \Gamma \setminus \{X_0\}$ . Define*

$$E_{\max} := \max_{X \in \Gamma} E[X] \quad \text{and} \quad B := \max_{X \hookrightarrow \alpha} \left| 1 - E[X] + \sum_{Y \in \Gamma} \#(Y)(\alpha) \cdot E[Y] \right|.$$

Then for all  $n \in \mathbb{N}$  with  $n \geq 2E[X_0]$  we have

$$\mathcal{P}(\mathbf{T}_{X_0} \geq n) \leq \exp \frac{2E[X_0] - n}{2B^2} \leq \exp \left( 1 - \frac{n}{8E_{max}^2} \right).$$

*Proof.* Let  $w \in \text{Run}(X_0)$ . We denote by  $I(w)$  the maximal number  $j \geq 0$  such that  $w(j-1) \neq \varepsilon$ . Given  $i \geq 0$ , we define  $m^{(i)}(w) := E[w(i)] + \min\{i, I(w)\}$ . We prove that  $E(m^{(i+1)} \mid m^{(i)}) = m^{(i)}$ , i.e.,  $m^{(0)}, m^{(1)}, \dots$  forms a martingale. It has been shown in [16] that

$$\begin{aligned} E[X] &= \sum_{X \xrightarrow{x} \varepsilon} x + \sum_{X \xrightarrow{x} Y} x \cdot (1 + E[Y]) + \sum_{X \xrightarrow{x} YZ} x \cdot (1 + E[Y] + E[Z]) \\ &= 1 + \sum_{X \xrightarrow{x} Y} x \cdot E[Y] + \sum_{X \xrightarrow{x} YZ} x \cdot (E[Y] + E[Z]). \end{aligned}$$

On the other hand, let us fix a path  $u \in \text{FPath}(X_0)$  of length  $i$  and let  $w$  be an arbitrary run of  $\text{Run}(u)$ . First assume that  $u(i-1) = X\alpha \in \Gamma\Gamma^*$ . Then we have:

$$\begin{aligned} &\mathbb{E} \left[ m^{(i+1)} \mid \text{Run}(u) \right] \\ &= \sum_{X \xrightarrow{x} \varepsilon} x \cdot (m^{(i)}(w) - E[X] + 1) + \sum_{X \xrightarrow{x} Y} x \cdot (m^{(i)}(w) - E[X] + E[Y] + 1) + \\ &\quad + \sum_{X \xrightarrow{x} YZ} x \cdot (m^{(i)}(w) - E[X] + E[Y] + E[Z] + 1) \\ &= m^{(i)}(w) - E[X] + 1 + \sum_{X \xrightarrow{x} Y} x \cdot E[Y] + \sum_{X \xrightarrow{x} YZ} x \cdot (E[Y] + E[Z]) \\ &= m^{(i)}(w) \end{aligned}$$

If  $u(i-1) = \varepsilon$ , then for every  $w \in \text{Run}(u)$  we have  $m^{(i+1)}(w) = I(w) = m^{(i)}(w)$ . This proves that  $m^{(0)}, m^{(1)}, \dots$  is a martingale.

By Azuma's inequality (see [29]), we have

$$\mathcal{P}(m^{(n)} - E[X_0] \geq n - E[X_0]) \leq \exp \left( \frac{-(n - E[X_0])^2}{2 \sum_{k=1}^n B^2} \right) \leq \exp \left( \frac{2E[X_0] - n}{2B^2} \right).$$

For every  $w \in \text{Run}(X_0)$  we have that  $w(n) \neq \varepsilon$  implies  $m^{(n)} \geq n$ . It follows:

$$\mathcal{P}(\mathbf{T}_{X_0} \geq n) \leq \mathcal{P}(m^{(n)} \geq n) \leq \exp \left( \frac{2E[X_0] - n}{2B^2} \right) \leq \exp \left( 1 - \frac{n}{8E_{max}^2} \right),$$

where the final inequality follows from the inequality  $B \leq 2E_{max}$ .  $\square$

The following proposition establishes the upper bound of Theorem 7 (3):

**Proposition 10.** *Let  $\Delta$  be an almost surely terminating pBPA with stack alphabet  $\Gamma$ . Assume that  $X_0$  depends on all  $X \in \Gamma \setminus \{X_0\}$ . Let  $p_{min} = \min\{p \mid X \xrightarrow{p} \alpha \text{ in } \Delta\}$ . Let  $h$  denote the height of the DAG of SCCs. Then there is  $n_0 \in \mathbb{N}$  such that*

$$\mathcal{P}(\mathbf{T}_{X_0} \geq n) \leq \frac{18h|\Gamma|/p_{min}^{3|\Gamma|}}{n^{1/(2^{h+1}-2)}} \quad \text{for all } n \geq n_0.$$

*Proof (sketch; a full proof is given in Section 6.2).* Assume that  $E[X_0]$  is infinite. To give some idea of the (quite involved) proof, let us first consider a simple pBPA  $\Delta$  with  $\Gamma = \{X\}$  and the rules  $X \hookrightarrow XX$  and  $X \hookrightarrow \varepsilon$ . In fact,  $\Delta$  is closely related to a simple random walk starting at 1, for which the time until it hits 0 can be exactly analyzed (see, e.g., [29]). Clearly, we have  $h = |\Gamma| = 1$  and  $p_{\min} = 1/2$ . Theorem 7(3) implies  $\mathcal{P}(\mathbf{T}_X \geq n) \in \mathcal{O}(1/\sqrt{n})$ . Let us sketch why this upper bound holds.

Let  $\theta > 0$ , define  $g(\theta) := \frac{1}{2} \cdot \exp(-\theta \cdot (-1)) + \frac{1}{2} \cdot \exp(-\theta \cdot (+1))$ , and define for a run  $w \in \text{Run}(X)$  the sequence

$$m_\theta^{(i)}(w) = \begin{cases} \exp(-\theta \cdot |w(i)|) / g(\theta)^i & \text{if } i = 0 \text{ or } w(i-1) \neq \varepsilon \\ m_\theta^{(i-1)}(w) & \text{otherwise.} \end{cases}$$

One can show (cf. [29]) that  $m_\theta^{(0)}, m_\theta^{(1)}, \dots$  is a martingale, i.e.,  $\mathbb{E}[m_\theta^{(i)} \mid m_\theta^{(i-1)}] = m_\theta^{(i-1)}$  for all  $\theta > 0$ . Our proof crucially depends on some analytic properties of the function  $g : \mathbb{R} \rightarrow \mathbb{R}$ : It is easy to verify that  $1 = g(0) < g(\theta)$  for all  $\theta > 0$ , and  $0 = g'(0)$ , and  $1 = g''(0)$ . One can show that Doob's Optional-Stopping Theorem (see Theorem 10.10 (ii) of [29]) applies, which implies  $m_\theta^{(0)} = \mathbb{E}[m_\theta^{(\mathbf{T}_X)}]$ . It follows that for all  $n \in \mathbb{N}$  and  $\theta > 0$  we have that

$$\begin{aligned} \exp(-\theta) = m_\theta^{(0)} &= \mathbb{E}[m_\theta^{(\mathbf{T}_X)}] = \mathbb{E}[g(\theta)^{-\mathbf{T}_X}] = \sum_{i=0}^{\infty} \mathcal{P}(\mathbf{T}_X = i) \cdot g(\theta)^{-i} \quad (1) \\ &\leq \sum_{i=0}^{n-1} \mathcal{P}(\mathbf{T}_X = i) \cdot 1 + \sum_{i=n}^{\infty} \mathcal{P}(\mathbf{T}_X = i) \cdot g(\theta)^{-n} \\ &= 1 - \mathcal{P}(\mathbf{T}_X \geq n) + \mathcal{P}(\mathbf{T}_X \geq n) \cdot g(\theta)^{-n} \end{aligned}$$

Rearranging this inequality yields  $\mathcal{P}(\mathbf{T}_X \geq n) \leq \frac{1 - \exp(-\theta)}{1 - g(\theta)^{-n}}$ , from which one obtains, setting  $\theta := 1/\sqrt{n}$ , and using the mentioned properties of  $g$  and several applications of l'Hopital's rule, that  $\mathcal{P}(\mathbf{T}_X \geq n) \in \mathcal{O}(1/\sqrt{n})$ .

Next we sketch how we generalize this proof to pBPA that consist of only one SCC, but have more than one stack symbol. In this case, the term  $|w(i)|$  in the definition of  $m_\theta^{(i)}(w)$  needs to be replaced by the sum of *weights* of the symbols in  $w(i)$ . Each  $Y \in \Gamma$  has a weight which is drawn from the dominant eigenvector of a certain matrix, which is characteristic for  $\Delta$ . Perron-Frobenius theory guarantees the existence of a suitable weight vector  $\mathbf{u} \in \mathbb{R}_+^\Gamma$ . The function  $g$  consequently needs to be replaced by a function  $g_Y$  for each  $Y \in \Gamma$ . We need to keep the property that  $g_Y''(0) > 0$ . Intuitively, this means that  $\Delta$  must have, for each  $Y \in \Gamma$ , a rule  $Y \hookrightarrow \alpha$  such that  $Y$  and  $\alpha$  have different weights. This can be accomplished by transforming  $\Delta$  into a certain normal form.

Finally, we sketch how the proof is generalized to pBPA with more than one SCC. For simplicity, assume that  $\Delta$  has only two stack symbols, say  $X$  and  $Y$ , where  $X$  depends on  $Y$ , but  $Y$  does not depend on  $X$ . Let us change the execution order of pBPA as follows: whenever a rule with  $\alpha \in \Gamma^*$  on the right hand side fires, then all  $X$ -symbols in  $\alpha$  are added on top of the stack, but all  $Y$ -symbols are added at the *bottom* of the stack. This change does not influence the termination time of pBPA, but

it allows to decompose runs into two phases: an  $X$ -phase where  $X$ -rules are executed which may produce  $Y$ -symbols or further  $X$ -symbols; and a  $Y$ -phase where  $Y$ -rules are executed which may produce further  $Y$ -symbols but no  $X$ -symbols, because  $Y$  does not depend on  $X$ . Arguing only qualitatively, assume that  $\mathbf{T}_X$  is “large”. Then either (a) the  $X$ -phase is “long” or (b) the  $X$ -phase is “short”, but the  $Y$ -phase is “long”. For the probability of event (a) one can give an upper bound using the bound for one SCC, because the produced  $Y$ -symbols can be ignored. For event (b), observe that if the  $X$ -phase is short, then only few  $Y$ -symbols can be created during the  $X$ -phase. For a bound on the probability of event (b) we need a bound on the probability that a pBPA with one SCC and a “short” initial configuration takes a “long” time to terminate. The previously sketched proof for an initial configuration with a single stack symbol can be suitably generalized to handle other “short” configurations. All details are given in Section 6.2.  $\square$

The following proposition establishes the lower bound of Theorem 7 (3):

**Proposition 11.** *Let  $\Delta$  be an almost surely terminating pBPA with stack alphabet  $\Gamma$ . Assume that  $X_0$  depends on all  $X \in \Gamma \setminus \{X_0\}$ . Assume  $E[X_0] = \infty$ . Then there is  $c > 0$  such that*

$$\frac{c}{\sqrt{n}} \leq \mathcal{P}(\mathbf{T}_{X_0} \geq n) \quad \text{for all } n \in \mathbb{N}.$$

The proof of Proposition 11 follows the lines of the previous proof sketch, but with an additional trick: To obtain the desired bound, one needs to take the derivative with respect to  $\theta$  on both sides of Equation (1). The full proof is given in Section 6.3.

**Tightness of the bounds in the case of infinite expectation.** If  $E[X_0]$  is infinite, the lower and upper bounds of Theorem 7 (3) asymptotically coincide in the “strongly connected” case (i.e., where  $h = 1$  holds for the height of the DAG of the SCCs of the dependence relation). In other words, in the strongly connected case we must have  $\mathcal{P}(\mathbf{T} \geq n) \in \Theta(1/\sqrt{n})$ . Otherwise (i.e., for larger  $h$ ) the upper bound in Theorem 7 (3) cannot be substantially tightened. This follows from the following proposition:

**Proposition 12.** *Let  $\Delta_h$  be the pBPA with  $\Gamma_h = \{X_1, \dots, X_h\}$  and the following rules:*

$$X_h \xrightarrow{1/2} X_h X_h, X_h \xrightarrow{1/2} X_{h-1}, \dots, X_2 \xrightarrow{1/2} X_2 X_2, X_2 \xrightarrow{1/2} X_1, X_1 \xrightarrow{1/2} X_1 X_1, X_1 \xrightarrow{1/2} \varepsilon$$

*Then  $[X_h] = 1$ ,  $E[X_h] = \infty$ , and there is  $c_h > 0$  with*

$$\frac{c_h}{n^{1/2^h}} \leq \mathcal{P}(\mathbf{T}_{X_h} \geq n) \quad \text{for all } n \in \mathbb{N}.$$

Proposition 12 is proved in Section 6.4.

## 5 Conclusions and Future Work

We have provided a reduction from stateful to stateless pPDA which gives new insights into the theory of pPDA and at the same time simplifies it substantially. We have used this reduction and martingale theory to exhibit a dichotomy result that precisely characterizes the distribution of the termination time in terms of its expected value.

Although the bounds presented in this paper are asymptotically optimal, there is still space for improvements. We conjecture that our results can be extended to more general reward-based models, where each configuration is assigned a nonnegative reward and the total reward accumulated in a given service is considered instead of its length. This is particularly challenging if the rewards are unbounded (for example, the reward assigned to a given configuration may correspond to the total memory allocated by the procedures in the current call stack). Full answers to these questions would generalize some of the existing deep results about simpler models, and probably reveal an even richer underlying theory of pPDA which is still undiscovered.

## 6 Proofs

In this section we give the missing proofs for the stated results. Some additional notation is used in the proofs.

- Given two sets  $K \subseteq \Sigma^*$  and  $L \subseteq \Sigma^* \cup \Sigma^\omega$ , we use  $K \cdot L$  (or just  $KL$ ) to denote the concatenation of  $K$  and  $L$ , i.e.,  $KL = \{ww' \mid w \in K, w' \in L\}$ .
- For a run  $w$  and  $i \in \mathbb{N}$ , we write  $w_i$  to denote the run  $w(i)w(i+1)\dots$ .

### 6.1 Proofs of Propositions 5 and 6

**Proposition 5.** *Let  $p_0X_0 \in Q \times \Gamma$  such that  $[p_0X_0\uparrow] = 1$ . Then there is a partial function  $\Upsilon : \text{Run}[M_\Delta](p_0X_0) \rightarrow \text{Run}[M_{\Delta_2}](\langle p_0X_0\uparrow \rangle)$  such that for every  $w \in \text{Run}[M_\Delta](p_0X_0)$ , where  $\Upsilon(w)$  is defined, and every  $n \in \mathbb{N}$  we have the following: if  $w(n) = qY\beta$ , then  $\Upsilon(w)(n) = \langle qY\uparrow \rangle\gamma$ , where  $\uparrow$  is either an element of  $Q$  or  $\uparrow$ . Further, for every measurable set of runs  $R \subseteq \text{Run}[M_{\Delta_2}](\langle p_0X_0\uparrow \rangle)$  we have that  $\Upsilon^{-1}(R)$  is measurable and  $\mathcal{P}(R) = \mathcal{P}(\Upsilon^{-1}(R))$ .*

*Proof.* Let  $w \in \text{Run}[M_\Delta](p_0X_0)$ . We define an infinite sequence  $\bar{w}$  over  $\bar{\Gamma}^*$  inductively as follows:

- $\bar{w}(0) = \langle p_0X_0\uparrow \rangle$
- If  $\bar{w}(i) = \varepsilon$  (which intuitively means that an “error” was indicated while defining the first  $i$  symbols of  $w$ ), then  $w(i+1) = \varepsilon$ . Now let us assume that  $\bar{w}(i) = \langle pX\uparrow \rangle\alpha$ , where  $\uparrow \in Q \cup \{\uparrow\}$ , and  $w(i) = pX\gamma$  for some  $\gamma \in \Gamma^*$ . Let  $pX \hookrightarrow r\beta$  be the rule of  $\Delta$  used to derive the transition  $w(i) \rightarrow w(i+1)$ . Then

$$\bar{w}(i+1) = \begin{cases} \alpha & \text{if } \beta = \varepsilon \text{ and } \uparrow = r; \\ \langle rY\uparrow \rangle\alpha & \text{if } \beta = Y \text{ and } [rY\uparrow] > 0; \\ \langle rYs \rangle \langle sZ\uparrow \rangle\alpha & \text{if } \beta = YZ, [sZ\uparrow] > 0, \text{ and there is } k > i \text{ such that } w(k) = sZ\gamma \text{ and } |w(j)| > |w(i)| \text{ for all } i < j < k; \\ \langle rY\uparrow \rangle\alpha & \text{if } \beta = YZ, [rY\uparrow] > 0, \text{ and } |w(j)| > |w(i)| \text{ for all } j > i; \\ \varepsilon & \text{otherwise.} \end{cases}$$

We say that  $w \in \text{Run}[M_\Delta](p_0X_0)$  is *valid* if  $\bar{w}(i) \neq \varepsilon$  for all  $i \in \mathbb{N}$ . One can easily check that if  $w$  is valid, then  $\bar{w}$  is a run of  $\bar{\Delta}$  initiated in  $\langle p_0X_0\uparrow \rangle$ . We put  $\Upsilon(w) = \bar{w}$  for all valid  $w \in \text{Run}[M_\Delta](p_0X_0)$ . For invalid runs,  $\Upsilon$  stays undefined.

It follows directly from the definition of  $\bar{w}$  that for every valid  $w \in \text{Run}[M_\Delta](p_0X_0)$  and every  $i \in \mathbb{N}$  we have that if  $w(i) = qY\beta$  then  $\bar{w}(i) = \langle qY\uparrow \rangle\gamma$ , where  $\uparrow \in Q \cup \{\uparrow\}$ .

Now we check that for every measurable set of runs  $R \subseteq \text{Run}[M_\Delta](\langle p_0X_0\uparrow \rangle)$  we have that  $\Upsilon^{-1}(R)$  is measurable and  $\mathcal{P}(R) = \mathcal{P}(\Upsilon^{-1}(R))$ . First, realize that the set of all *invalid*  $w \in \text{Run}[M_\Delta](p_0X_0)$  is measurable and its probability is zero. Hence, it suffices to show that for every finite path  $\bar{v}$  in  $M_{\bar{\Delta}}$  initiated in  $\langle p_0X_0\uparrow \rangle$  we have that  $\Upsilon^{-1}(\text{Run}[M_{\bar{\Delta}}](\bar{v}))$  is measurable and  $\mathcal{P}(\Upsilon^{-1}(\text{Run}[M_{\bar{\Delta}}](\bar{v}))) = \mathcal{P}(\text{Run}[M_{\bar{\Delta}}](\bar{v}))$ . For simplicity, we write just  $\Upsilon^{-1}(\bar{v})$  instead of  $\Upsilon^{-1}(\text{Run}[M_{\bar{\Delta}}](\bar{v}))$ .

Observe that every configuration  $\bar{\gamma}$  reachable from  $\langle p_0X_0\uparrow \rangle$  in  $M_{\bar{\Delta}}$  is of the form  $\bar{\gamma} = \langle p_1X_1p_2 \rangle \cdots \langle p_kX_kp_{k+1} \rangle \langle p_{k+1}Y\uparrow \rangle$  where  $k \geq 0$ . We put

$$P[\bar{\gamma}] = [p_1X_1p_2] \cdots [p_kX_kp_{k+1}] \cdot [p_{k+1}Y\uparrow]$$

Further, we say that a configuration  $p\alpha$  of  $\Delta$  is *compatible* with  $\bar{\gamma}$  if  $p = p_1$  and  $\alpha = X_1 \cdots X_kY\beta$  for some  $\beta \in \Gamma^*$ . A run  $w$  initiated in such a compatible configuration  $p_1X_1 \cdots X_kY\beta$  *models*  $\bar{\gamma}$ , written  $w \models \bar{\gamma}$ , if  $w$  is of the form

$$p_1X_1 \cdots X_kY\beta \rightarrow^* p_2X_2 \cdots X_kY\beta \rightarrow^* \cdots \rightarrow^* p_{k+1}Y\beta \rightarrow \cdots$$

where for all  $1 \leq i \leq k$ , the stack length of all intermediate configurations visited along the subpath  $p_iX_i \cdots X_kY\beta \rightarrow^* p_{i+1}X_{i+1} \cdots X_kY\beta$  is at least  $|X_i \cdots X_kY\beta|$ . Further, the stack length in all configurations visited after  $p_{k+1}Y\beta$  is at least  $|Y\beta|$ . A straightforward induction on  $k$  reveals that

$$\mathcal{P}\{w \in \text{Run}(p_1X_1 \cdots X_kY\beta) \mid w \models \bar{\gamma}\} = P[\bar{\gamma}] \quad (2)$$

Let  $\bar{v}\bar{\alpha}$ , where  $\bar{\alpha} \in \bar{\Gamma}^*$ , be a finite path in  $M_{\bar{\Delta}}$  initiated in  $\langle p_0X_0\uparrow \rangle$ , and let  $\mathcal{E}(\bar{v}\bar{\alpha})$  be the set of all finite path  $vA$  in  $M_\Delta$  initiated in  $p_0X_0$  such that  $A \in Q \times \Gamma^*$ ,  $|vA| = |\bar{v}\bar{\alpha}|$ , and  $\Upsilon^{-1}(\bar{v}\bar{\alpha})$  contains a run that starts with  $vA$ . One can easily check that if  $vA \in \mathcal{E}(\bar{v}\bar{\alpha})$ , then  $A$  is compatible with  $\bar{\alpha}$ . Further,

$$\Upsilon^{-1}(\bar{v}\bar{\alpha}) = \bigcup_{vA \in \mathcal{E}(\bar{v}\bar{\alpha})} vA \odot \{w \in \text{Run}[M_\Delta](A) \mid w \models \bar{\alpha}\} \quad (3)$$

From (3) we obtain that  $\Upsilon^{-1}(\bar{v}\bar{\alpha})$  is measurable, and by combining (2) and (3) we obtain

$$\mathcal{P}(\Upsilon^{-1}(\bar{v}\bar{\alpha})) = P[\bar{\alpha}] \cdot \sum_{vA \in \mathcal{E}(\bar{v}\bar{\alpha})} \mathcal{P}(\text{Run}(vA)) \quad (4)$$

Now we show that  $\mathcal{P}(\Upsilon^{-1}(\bar{v}\bar{\alpha})) = \mathcal{P}(\text{Run}(\bar{v}\bar{\alpha}))$ . We proceed by induction on  $|\bar{v}\bar{\alpha}|$ . The base case when  $\bar{v}\bar{\alpha} = \langle p_0X_0\uparrow \rangle$  is immediate. Now suppose that  $\bar{v}\bar{\alpha} = \bar{u}\bar{\beta}\bar{\alpha}$ , where  $\bar{\beta} \xrightarrow{x} \bar{\alpha}$ .

By applying (3) and (4) we obtain

$$\begin{aligned}
\mathcal{P}(\Upsilon^{-1}(\bar{u}\bar{\beta}\bar{\alpha})) &= \mathcal{P}\left(\bigcup_{uBA \in \mathcal{E}(\bar{u}\bar{\beta}\bar{\alpha})} uBA \odot \{w \in \text{Run}(A) \mid w \models \bar{\alpha}\}\right) \\
&= \mathcal{P}\left(\bigcup_{uB \in \mathcal{E}(\bar{u}\bar{\beta})} uB \odot \bigcup_{A \in Q \times \Gamma^*} \{w \in \text{Run}(BA) \mid uBA \in \mathcal{E}(\bar{u}\bar{\beta}\bar{\alpha}), w \models \bar{\beta}, w_1 \models \bar{\alpha}\}\right) \\
&= \sum_{uB \in \mathcal{E}(\bar{u}\bar{\beta})} \mathcal{P}(\text{Run}(uB)) \cdot \mathcal{P}\left(\bigcup_{A \in Q \times \Gamma^*} \{w \in \text{Run}(BA) \mid uBA \in \mathcal{E}(\bar{u}\bar{\beta}\bar{\alpha}), w \models \bar{\beta}, w_1 \models \bar{\alpha}\}\right) \\
&=^* \sum_{uB \in \mathcal{E}(\bar{u}\bar{\beta})} \mathcal{P}(\text{Run}(uB)) \cdot P[\bar{\beta}] \cdot x \\
&= x \cdot \mathcal{P}(\Upsilon^{-1}(\bar{u}\bar{\beta})) \\
&= \mathcal{P}(\text{Run}(\bar{u}\bar{\beta}\bar{\alpha}))
\end{aligned}$$

The (\*) equality is proved by case analysis (we distinguish possible forms of the rule which generates the transition  $\bar{\beta} \xrightarrow{x} \bar{\alpha}$ ).  $\square$

**Proposition 6.** *Let  $pXq \in Q \times \Gamma \times Q$  and  $[pXq] > 0$ . Then almost all runs of  $M_{\Delta_\bullet}$  initiated in  $\langle pXq \rangle$  terminate, i.e., reach  $\varepsilon$ . Further, for all  $n \in \mathbb{N}$  we have that*

$$\mathcal{P}(\mathbf{T}_{pX} = n \mid \text{Run}(pXq)) = \mathcal{P}(\mathbf{T}_{\langle pXq \rangle} = n \mid \text{Run}(\langle pXq \rangle))$$

*Proof.* For every  $n \in \mathbb{N}$  we define

$$\begin{aligned}
D_{pXq}(n) &:= \mathcal{P}(\text{Run}(pXq), \mathbf{T}_{pX} = n \mid \text{Run}(pX)) \\
D_{\langle pXq \rangle}(n) &:= \mathcal{P}(\mathbf{T}_{\langle pXq \rangle} = n \mid \text{Run}(\langle pXq \rangle))
\end{aligned}$$

We prove the following:

$$D_{pXq}(n) = [pXq] \cdot D_{\langle pXq \rangle}(n). \quad (5)$$

Notice that (5) implies  $\mathcal{P}(\mathbf{T}_{pX} = n \mid \text{Run}(pXq)) = \mathcal{P}(\mathbf{T}_{\langle pXq \rangle} = n \mid \text{Run}(\langle pXq \rangle))$ , as  $\mathcal{P}(\mathbf{T}_{pX} = n \mid \text{Run}(pXq)) = D_{pXq}(n)/[pXq]$ .

To prove (5), we proceed by induction on  $n$ . First, assume that  $n = 1$ . If  $pX \xrightarrow{x} q\varepsilon$ , then  $\langle pXq \rangle \xrightarrow{x} \varepsilon$ , where  $y = \frac{x}{[pXq]}$  and thus

$$D_{pXq}(1) = x = \frac{[pXq]x}{[pXq]} = [pXq]y = [pXq]D_{\langle pXq \rangle}(1).$$

If there is no rule  $pX \hookrightarrow q\varepsilon$  in  $\Delta$ , then there is no rule  $\langle pXq \rangle \hookrightarrow \varepsilon$  in  $\Delta_\bullet$ .



Assume that  $n > 1$ . Let us first prove that  $D_{pXq}(n)$  can be decomposed according to the first step:

$$D_{pXq}(n) = \sum_{pX \xrightarrow{x} rY} x \cdot D_{rYq}(n-1) + \sum_{i=1}^{n-1} \sum_{pX \xrightarrow{x} rYZ} \sum_{s \in Q} x \cdot D_{rYs}(i) \cdot D_{sZq}(n-i-1) \quad (6)$$

To prove (6) we introduce some notation. For every  $rYs \in Q \times \Gamma \times Q$  and  $i \in \mathbb{N}$  we denote by  $B_{rYs}(i)$  the set of all paths from  $rY$  to  $s\varepsilon$  of length  $i$ . We also denote by  $B_{rYs}(i) \downharpoonright Z$  the set of all paths of the form  $p_0\alpha_0Z \cdots p_i\alpha_iZ$  where  $p_0\alpha_0 \cdots p_i\alpha_i$  belongs to  $B_{rYs}(i)$ . We have

$$B_{pXq}(n) = \bigcup_{pX \hookrightarrow rY} B_{rYs}(n-1) \cup \bigcup_{i=1}^{n-1} \bigcup_{pX \xrightarrow{x} rYZ} \bigcup_{s \in Q} \{pX\} \cdot B_{rYs}(i) \downharpoonright Z \cdot B_{sZq}(n-i-1)$$

where all the unions are disjoint. Now the probability of following a path of  $B_{rYs}(i) \downharpoonright Z$  is equal to the probability of following a path of  $B_{rYs}(i)$ , which is  $D_{rYs}(i)$ . Thus we have that

$$\begin{aligned} \mathcal{P}(\text{Run}(\{pX\} \cdot B_{rYs}(i) \downharpoonright Z \cdot B_{sZq}(n-i-1))) &= x \cdot \mathcal{P}(B_{rYs}(i) \downharpoonright Z \cdot \text{Run}(B_{sZq}(n-i-1))) \\ &= x \cdot \mathcal{P}(\text{Run}(B_{rYs}(i)) \downharpoonright Z) \cdot \mathcal{P}(\text{Run}(B_{sZq}(n-i-1))) \\ &= x \cdot \mathcal{P}(\text{Run}(B_{rYs}(i))) \cdot D_{sZq}(n-i-1) \\ &= x \cdot D_{rYs}(i) \cdot D_{sZq}(n-i-1). \end{aligned}$$

It follows that

$$\begin{aligned} D_{pXq}(n) &= \mathcal{P}(\text{Run}(B_{pXq}(n))) \\ &= \mathcal{P}(\text{Run} \left( \bigcup_{pX \hookrightarrow rY} B_{rYs}(n-1) \cup \bigcup_{i=1}^{n-1} \bigcup_{pX \xrightarrow{x} rYZ} \bigcup_{s \in Q} \{pX\} \cdot B_{rYs}(i) \downharpoonright Z \cdot B_{sZq}(n-i-1) \right)) \\ &= \sum_{pX \xrightarrow{x} rY} x \cdot \mathcal{P}(\text{Run}(B_{rYs}(n-1))) + \\ &\quad + \sum_{i=1}^{n-1} \sum_{pX \xrightarrow{x} rYZ} \sum_{s \in Q} x \cdot \mathcal{P}(\text{Run}(B_{rYs}(i))) \cdot \mathcal{P}(\text{Run}(B_{sZq}(n-i-1))) \\ &= \sum_{pX \xrightarrow{x} rY} x \cdot D_{rYq}(n-1) + \sum_{i=1}^{n-1} \sum_{pX \xrightarrow{x} rYZ} \sum_{s \in Q} x \cdot D_{rYs}(i) \cdot D_{sZq}(n-i-1), \end{aligned}$$

which proves (6). Now we are ready to finish the induction proof of (5).

$$\begin{aligned}
D_{pXq}(n) &= \sum_{pX \xrightarrow{x} rY} x \cdot D_{rYq}(n-1) + \sum_{i=1}^{n-1} \sum_{pX \xrightarrow{x} rYZ} \sum_{s \in Q} x \cdot D_{rYs}(i) \cdot D_{sZq}(n-i-1) \\
&= \sum_{pX \xrightarrow{x} rY} x \cdot D_{\langle rYq \rangle}(n-1) \cdot [rYq] + \\
&\quad + \sum_{i=1}^{n-1} \sum_{pX \xrightarrow{x} rYZ} \sum_{s \in Q} x \cdot D_{\langle rYs \rangle}(i) \cdot [rYs] \cdot D_{\langle sZq \rangle}(n-i-1) \cdot [sZq] \\
&= [pXq] \cdot \left( \sum_{pX \xrightarrow{x} rY} \frac{x[rYq]}{[pXq]} \cdot D_{\langle rYq \rangle}(n-1) + \right. \\
&\quad \left. + \sum_{i=1}^{n-1} \sum_{pX \xrightarrow{x} rYZ} \sum_{s \in Q} \frac{x[rYs][sZq]}{[pXq]} \cdot D_{\langle rYs \rangle}(i) \cdot D_{\langle sZq \rangle}(n-i-1) \right) \\
&= [pXq] \cdot \left( \sum_{\langle pXq \rangle \xrightarrow{y} \langle rYq \rangle} y \cdot D_{\langle rYq \rangle}(n-1) + \right. \\
&\quad \left. + \sum_{i=1}^{n-1} \sum_{\langle pXq \rangle \xrightarrow{y} \langle rYs \rangle \langle sZq \rangle} y \cdot D_{\langle rYs \rangle}(i) \cdot D_{\langle sZq \rangle}(n-i-1) \right) \\
&= [pXq] \cdot D_{\langle pXq \rangle}(n)
\end{aligned}$$

Finally, observe that  $\sum_{n=1}^{\infty} D_{\langle pXq \rangle}$  is the probability of reaching  $\varepsilon$  from  $\langle pXq \rangle$  and that

$$\sum_{n=1}^{\infty} D_{\langle pXq \rangle} = \sum_{n=1}^{\infty} \frac{D_{pXq}(n)}{[pXq]} = \frac{1}{[pXq]} \cdot \sum_{n=1}^{\infty} D_{pXq}(n) = 1.$$

□

## 6.2 Proof of Proposition 10

In this subsection we prove Proposition 10. Given a finite set  $\Gamma$ , we regard the elements of  $\mathbb{R}^{\Gamma}$  as vectors. Given two vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{\Gamma}$ , we define a scalar product by setting  $\mathbf{u} \bullet \mathbf{v} := \sum_{X \in \Gamma} \mathbf{u}(X) \cdot \mathbf{v}(X)$ . Further, elements of  $\mathbb{R}^{\Gamma \times \Gamma}$  are regarded as matrices, with the usual matrix-vector multiplication.

It will be convenient for the proof to measure the termination time of pBPA starting in an arbitrary initial configuration  $\alpha_0 \in \Gamma\Gamma^*$ , not just with a single initial symbol  $X_0 \in \Gamma$ . To this end we generalize  $\mathbf{T}_{X_0}$ ,  $\text{Run}(X_0)$ , etc. to  $\mathbf{T}_{\alpha_0}$ ,  $\text{Run}(\alpha_0)$ , etc. in the straightforward way.

It will also be convenient to allow “pBPA” that have transition rules with more than two stack symbols on the right-hand side. We call them *relaxed pBPA*. All concepts associated to a pBPA, e.g., the induced Markov chain, termination time, etc., are defined analogously for relaxed pBPA.

A relaxed pBPA is called *strongly connected*, if the DAG of the dependence relation on its stack alphabet consists of a single SCC.

For any  $\alpha \in \Gamma^*$ , define  $\#(\alpha)$  as the Parikh image of  $\alpha$ , i.e., the vector of  $\mathbb{N}^\Gamma$  such that  $\#(\alpha)(Y)$  is the number of occurrences of  $Y$  in  $\alpha$ . Given a relaxed pBPA  $\Delta$ , let  $A_\Delta \in \mathbb{R}^{\Gamma \times \Gamma}$  be the matrix with

$$A_\Delta(X, Y) = \sum_{X \xrightarrow{p} \alpha} p \cdot \#(\alpha)(Y).$$

We drop the subscript of  $A_\Delta$  if  $\Delta$  is clear from the context. Intuitively,  $A(X, Y)$  is the expected number of  $Y$ -symbols pushed on the stack when executing a rule with  $X$  on the left hand side. For instance, if  $X \hookrightarrow XX$  and  $X \hookrightarrow \varepsilon$ , then  $A(X, X) = 2/5$ . Note that  $A$  is nonnegative. The matrix  $A$  plays a crucial role in the analysis of pPDA and related models (see e.g. [20]) and in the theory of branching processes [21]. We have the following lemma:

**Lemma 13.** *Let  $\Delta$  be an almost surely terminating, strongly connected pBPA. Then there is a positive vector  $\mathbf{u} \in \mathbb{R}_+^\Gamma$  such that  $A \cdot \mathbf{u} \leq \mathbf{u}$ , where  $\leq$  is meant componentwise. All such vectors  $\mathbf{u}$  satisfy  $\frac{\mathbf{u}_{\min}}{\mathbf{u}_{\max}} \geq p_{\min}^{|\Gamma|}$ , where  $p_{\min}$  denotes the least rule probability in  $\Delta$ , and  $\mathbf{u}_{\min}$  and  $\mathbf{u}_{\max}$  denote the least and the greatest component of  $\mathbf{u}$ , respectively.*

*Proof.* Let  $X, Y \in \Gamma$ . Since  $\Delta$  is strongly connected, there is a sequence  $X = X_1, X_2, \dots, X_n = Y$  with  $n \geq 1$  such that  $X_i$  depends directly on  $X_{i+1}$  for all  $1 \leq i \leq n-1$ . A straightforward induction on  $n$  shows that  $A^n(X, Y) \neq 0$ ; i.e.,  $A$  is *irreducible*. The assumption that  $\Delta$  is almost surely terminating implies that the spectral radius of  $A$  is less than or equal to one, see, e.g., Section 8.1 of [20]. Perron-Frobenius theory (see, e.g., [1]) then implies that there is a positive vector  $\mathbf{u} \in \mathbb{R}_+^\Gamma$  such that  $A \cdot \mathbf{u} \leq \mathbf{u}$ ; e.g., one can take for  $\mathbf{u}$  the dominant eigenvector of  $A$ .

Let  $A \cdot \mathbf{u} \leq \mathbf{u}$ . It remains to show that  $\frac{\mathbf{u}_{\min}}{\mathbf{u}_{\max}} \geq p_{\min}^{|\Gamma|}$ . The proof is essentially given in [14], we repeat it for convenience. W.l.o.g. let  $\Gamma = \{X_1, \dots, X_{|\Gamma|}\}$ . We write  $\mathbf{u}_i$  for  $\mathbf{u}(X_i)$ . W.l.o.g. let  $\mathbf{u}_1 = \mathbf{u}_{\max}$  and  $\mathbf{u}_{|\Gamma|} = \mathbf{u}_{\min}$ . Since  $\Delta$  is strongly connected, there is a sequence  $1 = r_1, r_2, \dots, r_q = |\Gamma|$  with  $q \leq |\Gamma|$  such that  $X_{r_j}$  depends on  $X_{r_{j+1}}$  for all  $j$ . We have

$$\frac{\mathbf{u}_{\min}}{\mathbf{u}_{\max}} = \frac{\mathbf{u}_{|\Gamma|}}{\mathbf{u}_1} = \frac{\mathbf{u}_{r_q}}{\mathbf{u}_{r_{q-1}}} \cdot \dots \cdot \frac{\mathbf{u}_{r_2}}{\mathbf{u}_{r_1}}.$$

By the pigeonhole principle there is  $j$  with  $2 \leq j \leq q$  such that

$$\frac{\mathbf{u}_{\min}}{\mathbf{u}_{\max}} \geq \left( \frac{\mathbf{u}_s}{\mathbf{u}_t} \right)^{q-1} \geq \left( \frac{\mathbf{u}_s}{\mathbf{u}_t} \right)^{|\Gamma|} \quad \text{where } s := r_j \text{ and } t := r_{j-1}. \quad (7)$$

We have  $A \cdot \mathbf{u} \leq \mathbf{u}$ , which implies  $A(X_s, X_t) \cdot \mathbf{u}_t \leq \mathbf{u}_s$  and so  $A(X_s, X_t) \leq \mathbf{u}_s / \mathbf{u}_t$ . On the other hand, since  $X_s$  depends on  $X_t$ , we clearly have  $p_{\min} \leq A(X_s, X_t)$ . Combining those inequalities with (7) yields  $\frac{\mathbf{u}_{\min}}{\mathbf{u}_{\max}} \geq (A(X_s, X_t))^{|\Gamma|} \geq p_{\min}^{|\Gamma|}$ .  $\square$

Given a relaxed pBPA  $\Delta$  and vector  $\mathbf{u} \in \mathbb{R}_+^\Gamma$ , we say that  $\Delta$  is  $\mathbf{u}$ -progressive, if  $\Delta$  has, for all  $X \in \Gamma$ , a rule  $X \hookrightarrow \alpha$  such that  $|\mathbf{u}(X) - \#(\alpha) \cdot \mathbf{u}| \geq \mathbf{u}_{\min}/2$ . The following lemma states that, intuitively, any pBPA can be transformed into a  $\mathbf{u}$ -progressive relaxed pBPA that is at least as fast but no more than  $|\Gamma|$  times faster.

**Lemma 14.** Let  $\Delta$  be an almost surely terminating pBPA with stack alphabet  $\Gamma$ . Let  $p_{\min}$  denote the least rule probability in  $\Delta$ , and let  $\mathbf{u} \in \mathbb{R}_+^\Gamma$  with  $A_\Delta \cdot \mathbf{u} \leq \mathbf{u}$ . Then one can construct a  $\mathbf{u}$ -progressive, almost surely terminating relaxed pBPA  $\Delta'$  with stack alphabet  $\Gamma$  such that for all  $\alpha_0 \in \Gamma^*$  and for all  $a \geq 0$

$$\mathcal{P}'(\mathbf{T}_{\alpha_0} \geq a) \leq \mathcal{P}(\mathbf{T}_{\alpha_0} \geq a) \leq \mathcal{P}'(\mathbf{T}_{\alpha_0} \geq a/|\Gamma|),$$

where  $\mathcal{P}$  and  $\mathcal{P}'$  are the probability measures associated with  $\Delta$  and  $\Delta'$ , respectively. Furthermore, the least rule probability in  $\Delta'$  is at least  $p_{\min}^{|\Gamma|}$ , and  $A_{\Delta'} \cdot \mathbf{u} \leq \mathbf{u}$ . Finally, if  $A_\Delta \cdot \mathbf{u} = \mathbf{u}$ , then  $A_{\Delta'} \cdot \mathbf{u} = \mathbf{u}$ .

*Proof.* A sequence of transitions  $X_1 \hookrightarrow \alpha_1, \dots, X_n \hookrightarrow \alpha_n$  is called *derivation sequence* from  $X_1$  to  $\alpha_n$ , if for all  $i \in \{2, \dots, n\}$  the symbol  $X_i \in \Gamma$  occurs in  $\alpha_{i-1}$ . The word induced by a derivation sequence  $X_1 \hookrightarrow \alpha_1, \dots, X_n \hookrightarrow \alpha_n$  is obtained by taking  $\alpha_1$ , replacing an occurrence of  $X_2$  by  $\alpha_2$ , then replacing an occurrence of  $X_3$  by  $\alpha_3$ , etc., and finally replacing an occurrence of  $X_n$  by  $\alpha_n$ .

Given a pBPA  $\Delta$  and a derivation sequence  $s = (X_1 \xrightarrow{p_1} \alpha_1^1 X_2 \alpha_1^2, X_2 \xrightarrow{p_2} \alpha_2, \dots, X_n \xrightarrow{p_n} \alpha_n)$  with  $X_i \neq X_j$  for all  $1 \leq i < j \leq n$ , we define the *contraction*  $\text{Con}(s)$  of  $s$ , a set of  $X_1$ -transitions with possibly more than two symbols on the right hand side. The contraction  $\text{Con}(s)$  will include a rule  $X_1 \hookrightarrow \gamma$ , where  $\gamma$  is the word induced by  $s$ . We define  $\text{Con}(s)$  inductively over the length  $n$  of  $s$ . If  $n = 1$ , then  $\text{Con}(s) = \{X_1 \xrightarrow{p_1} \alpha_1^1 X_2 \alpha_1^2\}$ . If  $n \geq 2$ , let  $s' = (X_2 \xrightarrow{p_2} \alpha_2, \dots, X_n \xrightarrow{p_n} \alpha_n)$  and define

$$\delta_2 := \{X_2 \hookrightarrow \beta \mid X_2 \hookrightarrow \beta \text{ is a rule in } \Delta\} - \left\{X_2 \xrightarrow{p_2} \alpha_2\right\} \cup \text{Con}(s'); \quad (8)$$

i.e.,  $\delta_2$  is the set of  $X_2$ -transitions in  $\Delta$  with  $X_2 \xrightarrow{p_2} \alpha_2$  replaced by  $\text{Con}(s')$ . W.l.o.g. assume  $\delta_2 = \{X_2 \xrightarrow{q_1} \beta_1, \dots, X_2 \xrightarrow{q_k} \beta_k\}$ . Then we define

$$\text{Con}(s) := \left\{X_1 \xrightarrow{p_1 q_1} \alpha_1^1 \beta_1 \alpha_1^2, \dots, X_1 \xrightarrow{p_1 q_k} \alpha_1^1 \beta_k \alpha_1^2\right\}.$$

The following properties are easy to show by induction on  $n$ :

- (a)  $\text{Con}(s)$  contains  $X_1 \hookrightarrow \gamma$ , where  $\gamma$  is the word induced by  $s$ .
- (b) The rule probabilities are at least  $p_{\min}^n$ .
- (c) Let  $\Delta'$  be the relaxed pBPA obtained from  $\Delta$  by replacing  $X_1 \xrightarrow{p_1} \alpha_1^1 X_2 \alpha_1^2$  with  $\text{Con}(s)$ . Then each path in  $M_{\Delta'}$  corresponds in a straightforward way to a path in  $M_\Delta$ , namely to the path obtained by “re-expanding” the contractions. The corresponding path in  $M_\Delta$  has the same probability and is not shorter but at most  $|\Gamma|$  times longer than the one in  $M_{\Delta'}$ .
- (d) Let  $\Delta'$  be as in (c). Then  $A_{\Delta'} \cdot \mathbf{u} \leq \mathbf{u}$ . Let us prove that explicitly. The induction hypothesis  $n = 1$  is trivial. For the induction step, using the definition for  $\delta_2$  in (8) and  $\delta_2 = \{X_2 \xrightarrow{q_1} \beta_1, \dots, X_2 \xrightarrow{q_k} \beta_k\}$ , we know by the induction hypothesis that  $\sum_{i=1}^k q_i \cdot \#(\beta_i) \cdot \mathbf{u} \leq \mathbf{u}(X_2)$ . This implies

$$\sum_{i=1}^k p_1 q_i \cdot \#(\alpha_1^1 \beta_i \alpha_1^2) \cdot \mathbf{u} \leq p_1 \cdot \#(\alpha_1^1 X_2 \alpha_1^2) \cdot \mathbf{u}, \quad \text{and hence}$$

$$(A_{\Delta'} \cdot \mathbf{u})(X_1) \leq (A_\Delta \cdot \mathbf{u})(X_1) \leq \mathbf{u}(X_1).$$

Since  $A_\Delta$  and  $A_{\Delta'}$  may differ only in the  $X_1$ -row, we have  $A_{\Delta'} \cdot \mathbf{u} \leq \mathbf{u}$ .

- (e) Let  $\Delta'$  be as in (c) and (d). If  $A_\Delta \cdot \mathbf{u} = \mathbf{u}$ , then  $A_{\Delta'} \cdot \mathbf{u} = \mathbf{u}$ . This follows as in (d), with the inequality signs replaced by equality.

Associate to each symbol  $X_1 \in \Gamma$  a shortest derivation sequence

$$c(X_1) = (X_1 \hookrightarrow \alpha_1, \dots, X_{n-1} \hookrightarrow \alpha_{n-1}, X_n \hookrightarrow \varepsilon)$$

from  $X_1$  to  $\varepsilon$ . Since  $\Delta$  is almost surely terminating, the length of  $c(X_1)$  is at most  $|\Gamma|$  for all  $X_1 \in \Gamma$ . Let  $X_1 \in \Gamma$ , and let  $\gamma_1$  denote the word induced by  $c(X_1)$ , and let  $\gamma_2$  denote the word induced by the derivation sequence  $c_2(X_1) := (X_1 \hookrightarrow \alpha_1, \dots, X_{n-1} \hookrightarrow \alpha_{n-1})$ . We have  $\#(\gamma_2) \cdot \mathbf{u} = \#(\gamma_1) \cdot \mathbf{u} + \mathbf{u}(X_n) \geq \#(\gamma_1) \cdot \mathbf{u} + \mathbf{u}_{\min}$ , so we can choose  $\gamma \in \{\gamma_1, \gamma_2\}$  such that  $|\mathbf{u}(X_1) - \#(\gamma) \cdot \mathbf{u}| \geq \mathbf{u}_{\min}/2$ . Choose  $\hat{c}(X_1) \in \{c(X_1), c_2(X_1)\}$  such that  $\hat{c}(X_1)$  induces  $\gamma$ . (Of course, if  $c_2(X_1)$  has length zero, take  $\hat{c}(X_1) = c(X_1)$ .) Note that  $(X_1 \hookrightarrow \gamma) \in \text{Con}(\hat{c}(X_1))$ .

The relaxed pBPA  $\Delta'$  from the statement of the lemma is obtained by replacing, for all  $X_1 \in \Gamma$ , the first rule of  $\hat{c}(X_1)$  with  $\text{Con}(\hat{c}(X_1))$ . The properties (a)–(e) from above imply:

- (a) The relaxed pBPA  $\Delta'$  is  $\mathbf{u}$ -progressive.
- (b) The rule probabilities are at least  $p_{\min}^{|\Gamma|}$ .
- (c) For each finite path  $w'$  in  $M_{\Delta'}$  from some  $\alpha_0 \in \Gamma^*$  to  $\varepsilon$  there is a finite path  $w$  in  $M_\Delta$  from  $\alpha_0$  to  $\varepsilon$  such that  $|w'| \leq |w| \leq |\Gamma| \cdot |w'|$  and  $\mathcal{P}'(w') = \mathcal{P}(w)$ . Hence,  $\mathcal{P}'(\mathbf{T}_{\alpha_0} < a/|\Gamma|) \leq \mathcal{P}(\mathbf{T}_{\alpha_0} < a) \leq \mathcal{P}'(\mathbf{T}_{\alpha_0} < a)$  holds for all  $a \geq 0$ , which implies  $\mathcal{P}'(\mathbf{T}_{\alpha_0} \geq a) \leq \mathcal{P}(\mathbf{T}_{\alpha_0} \geq a) \leq \mathcal{P}'(\mathbf{T}_{\alpha_0} \geq a/|\Gamma|)$ .
- (d) We have  $A_{\Delta'} \cdot \mathbf{u} \leq \mathbf{u}$ .
- (e) If  $A_\Delta \cdot \mathbf{u} = \mathbf{u}$ , then  $A_{\Delta'} \cdot \mathbf{u} = \mathbf{u}$ .

This completes the proof of the lemma.  $\square$

**Proposition 15.** *Let  $\Delta$  be an almost surely terminating relaxed pBPA with stack alphabet  $\Gamma$ . Let  $\mathbf{u} \in \mathbb{R}_+^\Gamma$  be such that  $\mathbf{u}_{\max} = 1$  and  $A_\Delta \cdot \mathbf{u} \leq \mathbf{u}$  and  $\Delta$  is  $\mathbf{u}$ -progressive. Let  $p_{\min}$  denote the least rule probability in  $\Delta$ . Let  $C := 17|\Gamma|/(p_{\min} \cdot \mathbf{u}_{\min}^2)$ . Then for each  $k \in \mathbb{N}_0$  there is  $n_0 \in \mathbb{N}$  such that*

$$\mathcal{P}(\mathbf{T}_{\alpha_0} \geq n^{2k+2}/(2|\Gamma|)) \leq C/n \quad \text{for all } n \geq n_0 \text{ and for all } \alpha_0 \in \Gamma^* \text{ with } 1 \leq |\alpha_0| \leq n^k.$$

*Proof.* For each  $X \in \Gamma$  we define a function  $g_X : \mathbb{R} \rightarrow \mathbb{R}$  by setting

$$g_X(\theta) := \sum_{X \xrightarrow{p} \alpha} p \cdot \exp(-\theta \cdot (-\mathbf{u}(X) + \#(\alpha) \cdot \mathbf{u})).$$

The following lemma states important properties of  $g_X$ .

**Lemma 16.** *The following holds for all  $X \in \Gamma$ :*

- (a) For all  $\theta > 0$  we have  $1 = g_X(0) < g_X(\theta)$ .
- (b) For all  $\theta > 0$  we have  $0 \leq g'_X(0) < g'_X(\theta)$ .
- (c) For all  $\theta \geq 0$  we have  $0 < g''_X(\theta)$ . In particular,  $g''_X(0) \geq p_{\min} \cdot \mathbf{u}_{\min}^2/4$ .

*Proof (Proof of the lemma).*

- (a) Clearly,  $g_X(0) = 1$ . The inequality  $g_X(0) < g_X(\theta)$  follows from (b).  
(b) We have:

$$\begin{aligned} g_X(\theta) &= \sum_{X \xrightarrow{p} \alpha} p \cdot \exp(-\theta \cdot (-\mathbf{u}(X) + \#(\alpha) \cdot \mathbf{u})) \\ g'_X(\theta) &= \sum_{X \xrightarrow{p} \alpha} p \cdot (\mathbf{u}(X) - \#(\alpha) \cdot \mathbf{u}) \cdot \exp(-\theta \cdot (-\mathbf{u}(X) + \#(\alpha) \cdot \mathbf{u})) \end{aligned}$$

Let  $A(X)$  denote the  $X$ -row of  $A$ , i.e., the vector  $\mathbf{v} \in \mathbb{R}^F$  such that  $\mathbf{v}(Y) = A(X, Y)$ . Then  $A \cdot \mathbf{u} \leq \mathbf{u}$  implies

$$\begin{aligned} g'_X(0) &= \sum_{X \xrightarrow{p} \alpha} p \cdot (\mathbf{u}(X) - \#(\alpha) \cdot \mathbf{u}) \\ &= \mathbf{u}(X) - \sum_{X \xrightarrow{p} \alpha} p \cdot \#(\alpha) \cdot \mathbf{u} = \mathbf{u}(X) - A(X) \cdot \mathbf{u} \\ &\geq \mathbf{u}(X) - \mathbf{u}(X) = 0. \end{aligned}$$

The inequality  $g'_X(0) < g'_X(\theta)$  follows from (c).

- (c) We have

$$g''_X(\theta) = \sum_{X \xrightarrow{p} \alpha} p \cdot (\mathbf{u}(X) - \#(\alpha) \cdot \mathbf{u})^2 \cdot \exp(-\theta \cdot (-\mathbf{u}(X) + \#(\alpha) \cdot \mathbf{u})) > 0.$$

Since  $\Delta$  is  $\mathbf{u}$ -progressive, there is a rule  $X \xrightarrow{p} \alpha$  with  $|\mathbf{u}(X) - \#(\alpha) \cdot \mathbf{u}| \geq \mathbf{u}_{\min}/2$ . Hence, for  $\theta = 0$  we have  $g''_X(0) \geq p_{\min} \cdot \mathbf{u}_{\min}^2/4$ .

This proves the lemma. □

Let in the following  $\theta > 0$ . Given a run  $w \in \text{Run}(\alpha_0)$  and  $i \geq 0$ , we write  $X^{(i)}(w)$  for the symbol  $X \in \Gamma$  for which  $w(i) = X\alpha$ . Define

$$m_\theta^{(i)}(w) = \begin{cases} \exp(-\theta \cdot \#(w(i)) \cdot \mathbf{u}) \cdot \prod_{j=0}^{i-1} \frac{1}{g_{X^{(j)}(w)}(\theta)} & \text{if } i = 0 \text{ or } w(i-1) \neq \varepsilon \\ m_\theta^{(i-1)}(w) & \text{otherwise} \end{cases}$$

**Lemma 17.**  $m_\theta^{(0)}, m_\theta^{(1)}, \dots$  is a martingale.

*Proof (Proof of the lemma).* Let us fix a path  $v \in \text{FPath}(\alpha_0)$  of length  $i \geq 1$  and let  $w$  be an arbitrary run of  $\text{Run}(v)$ . First assume that  $v(i-1) = X\alpha \in \Gamma\Gamma^*$ . Then we

have:

$$\begin{aligned}
& \mathbb{E} \left[ m_{\theta}^{(i)} \mid Run(v) \right] \\
&= \mathbb{E} \left[ \exp(-\theta \cdot \#(w(i)) \cdot \mathbf{u}) \cdot \prod_{j=0}^{i-1} \frac{1}{g_{X^{(j)}(w)}(\theta)} \mid Run(v) \right] \\
&= \sum_{X \xrightarrow{p} \alpha} p \cdot \exp(-\theta \cdot (\#(w(i-1)) - \mathbf{1}_X + \#(\alpha)) \cdot \mathbf{u}) \cdot \prod_{j=0}^{i-1} \frac{1}{g_{X^{(j)}(w)}(\theta)} \\
&= \sum_{X \xrightarrow{p} \alpha} p \cdot \exp(-\theta \cdot (\#(w(i-1)) \cdot \mathbf{u} - \mathbf{u}(X) + \#(\alpha) \cdot \mathbf{u})) \cdot \prod_{j=0}^{i-1} \frac{1}{g_{X^{(j)}(w)}(\theta)} \\
&= \exp(-\theta \cdot \#(w(i-1)) \cdot \mathbf{u}) \cdot \sum_{X \xrightarrow{p} \alpha} p \cdot \exp(-\theta \cdot (-\mathbf{u}(X) + \#(\alpha) \cdot \mathbf{u})) \cdot \prod_{j=0}^{i-1} \frac{1}{g_{X^{(j)}(w)}(\theta)} \\
&= \exp(-\theta \cdot \#(w(i-1)) \cdot \mathbf{u}) \cdot g_{X^{(i-1)}(w)}(\theta) \cdot \prod_{j=0}^{i-1} \frac{1}{g_{X^{(j)}(w)}(\theta)} \\
&= \exp(-\theta \cdot \#(w(i-1)) \cdot \mathbf{u}) \cdot \prod_{j=0}^{i-2} \frac{1}{g_{X^{(j)}(w)}(\theta)} \\
&= m_{\theta}^{(i-1)}(w).
\end{aligned}$$

If  $v(i-1) = \varepsilon$ , then for every  $w \in Run(v)$  we have  $m_{\theta}^{(i)}(w) = m_{\theta}^{(i-1)}(w)$ . Hence,  $m_{\theta}^{(0)}, m_{\theta}^{(1)}, \dots$  is a martingale.  $\square$

Since  $\theta > 0$  and since  $g_{X^{(j)}(w)}(\theta) \geq 1$  by Lemma 16(a), we have  $0 \leq m_{\theta}^{(i)}(w) \leq 1$ , so the martingale is bounded. Since, furthermore,  $\mathbf{T}_{\alpha_0}$  (we write only  $\mathbf{T}$  in the following) is finite with probability 1, it follows using Doob's Optional-Stopping Theorem (see

Theorem 10.10 (ii) of [29]) that  $m_\theta^{(0)} = \mathbb{E} \left[ m_\theta^{(\mathbf{T})} \right]$ . Hence we have for each  $n \in \mathbb{N}$ :

$$\begin{aligned}
& \exp(-\theta \cdot \mathbf{u}_{max} \cdot n^k) \\
& \leq \exp(-\theta \cdot \mathbf{u} \cdot \#(\alpha_0)) = m_\theta^{(0)} \\
& = \mathbb{E} \left[ m_\theta^{(\mathbf{T})} \right] \quad \text{(by optional-stopping)} \\
& = \mathbb{E} \left[ \exp(-\theta \cdot 0) \cdot \prod_{j=0}^{\mathbf{T}-1} \frac{1}{g_{X^{(j)}}(\theta)} \right] \\
& = \mathbb{E} \left[ \prod_{j=0}^{\mathbf{T}-1} \frac{1}{g_{X^{(j)}}(\theta)} \right] \\
& \leq \mathbb{E} \left[ \frac{1}{g_X(\theta)^{\mathbf{T}}} \right] \quad \text{(for some } X \in \Gamma) \\
& = \sum_{i=0}^{\infty} \mathcal{P}(\mathbf{T} = i) \cdot \frac{1}{g_X(\theta)^i} \\
& \leq \sum_{i=0}^{\lceil n^{2k+2}/(2|\Gamma|) \rceil - 1} \mathcal{P}(\mathbf{T} = i) \cdot 1 \quad \text{(Lemma 16 (a))} \\
& \quad + \sum_{i=\lceil n^{2k+2}/(2|\Gamma|) \rceil}^{\infty} \mathcal{P}(\mathbf{T} = i) \cdot \frac{1}{g_X(\theta)^{n^{2k+2}/(2|\Gamma|)}} \\
& = 1 - \mathcal{P}(\mathbf{T} \geq n^{2k+2}/(2|\Gamma|)) \\
& \quad + \mathcal{P}(\mathbf{T} \geq n^{2k+2}/(2|\Gamma|)) \cdot \frac{1}{g_X(\theta)^{n^{2k+2}/(2|\Gamma|)}}
\end{aligned}$$

Rearranging the inequality, we obtain

$$\mathcal{P}(\mathbf{T} \geq n^{2k+2}/(2|\Gamma|)) \leq \frac{1 - \exp(-\theta \cdot \mathbf{u}_{max} \cdot n^k)}{1 - g_X(\theta)^{-n^{2k+2}/(2|\Gamma|)}}. \quad (9)$$

For the following we set  $\theta = n^{-(k+1)}$ . We want to give an upper bound for the right hand side of (9). To this end we will show:

$$\lim_{n \rightarrow \infty} \frac{(1 - \exp(-n^{-(k+1)} \cdot \mathbf{u}_{max} \cdot n^k)) \cdot n}{1 - g_X(n^{-(k+1)})^{-n^{2(k+1)}/(2|\Gamma|)}} \leq \frac{1}{1 - \exp(-p_{min} \cdot \mathbf{u}_{min}^2/(16|\Gamma|))}. \quad (10)$$

Combining (9) with (10), we obtain

$$\begin{aligned}
\limsup_{n \rightarrow \infty} n \cdot \mathcal{P}(\mathbf{T} \geq n^{2k+2}/(2|\Gamma|)) & \leq \frac{1}{1 - \exp(-p_{min} \cdot \mathbf{u}_{min}^2/(16|\Gamma|))} \\
& < \frac{1}{1 - \left(1 - \frac{16}{17} \cdot (p_{min} \cdot \mathbf{u}_{min}^2/(16|\Gamma|))\right)} \\
& = 17|\Gamma|/(p_{min} \cdot \mathbf{u}_{min}^2),
\end{aligned}$$



which implies the proposition.

To prove (10), we compute limits for the nominator and the denominator separately. For the nominator, we use l'Hopital's rule to obtain:

$$\lim_{n \rightarrow \infty} \frac{1 - \exp(-\mathbf{u}_{max} \cdot n^{-1})}{n^{-1}} = \lim_{n \rightarrow \infty} \frac{-\mathbf{u}_{max} \cdot n^{-2} \cdot \exp(-\mathbf{u}_{max} \cdot n^{-1})}{-n^{-2}} = \mathbf{u}_{max} = 1.$$

For the denominator of (10) we consider first the following limit:

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{2|I|} \cdot n^{2(k+1)} \cdot \ln g_X(n^{-(k+1)}) \\ &= \frac{1}{2|I|} \lim_{n \rightarrow \infty} \frac{\ln g_X(n^{-(k+1)})}{n^{-2(k+1)}} \\ &= \frac{1}{2|I|} \lim_{n \rightarrow \infty} \frac{g'_X(n^{-(k+1)}) \cdot (-(k+1)) \cdot n^{-k-2}}{g_X(n^{-(k+1)}) \cdot (-2(k+1)) \cdot n^{-2k-3}} \quad (\text{l'Hopital's rule}) \\ &= \frac{1}{4|I|} \lim_{n \rightarrow \infty} \frac{g'_X(n^{-(k+1)})}{n^{-(k+1)}} \quad (\text{by Lemma 16 (a)}). \end{aligned}$$

If  $g'_X(0) > 0$ , then the limit is  $+\infty$ . Otherwise, by Lemma 16 (b), we have  $g'_X(0) = 0$  and hence

$$\begin{aligned} &= \frac{1}{4|I|} \lim_{n \rightarrow \infty} \frac{g''_X(n^{-(k+1)}) \cdot (-(k+1)) \cdot n^{-k-2}}{(-(k+1)) \cdot n^{-k-2}} \quad (\text{l'Hopital's rule}) \\ &= \frac{1}{4|I|} g''_X(0) \geq p_{min} \cdot \mathbf{u}_{min}^2 / (16|I|) \quad (\text{by Lemma 16 (c)}). \end{aligned}$$

This proves (10) and thus completes the proof of Proposition 15.  $\square$

The following lemma serves as induction base for the proof of Proposition 10.

**Lemma 18.** *Let  $\Delta$  be an almost surely terminating pBPA with stack alphabet  $\Gamma$ . Assume that all SCCs of  $\Delta$  are bottom SCCs. Let  $p_{min}$  denote the least rule probability in  $\Delta$ . Let  $D := 17|I|/p_{min}^{3|I|}$ . Then for each  $k \in \mathbb{N}_0$  there is  $n_0 \in \mathbb{N}$  such that*

$$\mathcal{P}(\mathbf{T}_{\alpha_0} \geq n^{2k+2}/2) \leq D/n \quad \text{for all } n \geq n_0 \text{ and for all } \alpha_0 \in \Gamma^* \text{ with } 1 \leq |\alpha_0| \leq n^k.$$

*Proof.* Decompose  $\Gamma$  into its SCCs, say  $\Gamma = \Gamma_1 \cup \dots \cup \Gamma_s$ , and let the pBPA  $\Delta_i$  be obtained by restricting  $\Delta$  to the  $\Gamma_i$ -symbols. For each  $i \in \{1, \dots, s\}$ , Lemma 13 gives a vector  $\mathbf{u}_i \in \mathbb{R}_+^{\Gamma_i}$ . W.l.o.g. we can assume for each  $i$  that the largest component of  $\mathbf{u}_i$  is equal to 1, because  $\mathbf{u}_i$  can be multiplied with any positive scalar without changing the properties guaranteed by Lemma 13. If the vectors  $\mathbf{u}_i$  are assembled (in the obvious way) to the vector  $\mathbf{u} \in \mathbb{R}_+^{\Gamma}$ , the assertions of Lemma 13 carry over; i.e., we have  $A_\Delta \cdot \mathbf{u} \leq \mathbf{u}$  and  $\mathbf{u}_{max} = 1$  and  $\mathbf{u}_{min} \geq p_{min}^{|I|}$ . Let  $\Delta'$  be the  $\mathbf{u}$ -progressive relaxed pBPA from Lemma 14, and denote by  $\mathcal{P}'$  and  $p'_{min}$  its associated probability measure and least rule probability, respectively. Then we have:

$$\begin{aligned} \mathcal{P}(\mathbf{T}_{\alpha_0} \geq n^{2k+2}/2) &\leq \mathcal{P}'(\mathbf{T}_{\alpha_0} \geq n^{2k+2}/(2|I|)) && (\text{by Lemma 14}) \\ &\leq 17|I|/(p'_{min} \cdot \mathbf{u}_{min}^2 \cdot n) && (\text{by Proposition 15}) \\ &\leq 17|I|/(p'_{min} \cdot p_{min}^{2|I|} \cdot n) && (\text{as argued above}) \\ &\leq 17|I|/(p_{min}^{3|I|} \cdot n) && (\text{by Lemma 14}). \end{aligned}$$

□

Now we are ready to prove Proposition 10, which is restated here.

**Proposition 10.** *Let  $\Delta$  be an almost surely terminating pBPA with stack alphabet  $\Gamma$ . Assume that  $X_0$  depends on all  $X \in \Gamma \setminus \{X_0\}$ . Let  $p_{\min} = \min\{p \mid X \xrightarrow{p} \alpha \text{ in } \Delta\}$ . Let  $h$  denote the height of the DAG of SCCs. Then there is  $n_0 \in \mathbb{N}$  such that*

$$\mathcal{P}(\mathbf{T}_{X_0} \geq n) \leq \frac{18h|\Gamma|/p_{\min}^{3|\Gamma|}}{n^{1/(2^{h+1}-2)}} \quad \text{for all } n \geq n_0.$$

*Proof.* Let  $D$  be the  $D$  from Lemma 18. We show by induction on  $h$ :

$$\mathcal{P}(\mathbf{T}_{X_0} \geq n^{2^{h+1}-2}) \leq \frac{hD}{n} \quad \text{for almost all } n \in \mathbb{N}. \quad (11)$$

Note that (11) implies the proposition. The case  $h = 1$  (induction base) is implied by Lemma 18. Let  $h \geq 2$ . Partition  $\Gamma$  into  $\Gamma_{\text{high}} \cup \Gamma_{\text{low}}$  such that  $\Gamma_{\text{low}}$  contains the variables of the SCCs of depth  $h$  in the DAG of SCCs, and  $\Gamma_{\text{high}}$  contains the other variables (in “higher” SCCs). If  $X_0 \in \Gamma_{\text{low}}$ , then we can restrict  $\Delta$  to the variables that are in the same SCC as  $X_0$ , and Lemma 18 implies (11). So we can assume  $X_0 \in \Gamma_{\text{high}}$ .

Assume for a moment that  $\mathcal{P}(\mathbf{T}_{X_0} \geq n^{2^{h+1}-2})$  holds for a run  $w \in \text{Run}(X_0)$ ; i.e., we have:

$$\begin{aligned} n^{2^{h+1}-2} &\leq |\{i \in \mathbb{N}_0 \mid w(i) \in \Gamma\Gamma^*\}| \\ &= |\{i \in \mathbb{N}_0 \mid w(i) \in \Gamma_{\text{high}}\Gamma^*\}| + |\{i \in \mathbb{N}_0 \mid w(i) \in \Gamma_{\text{low}}\Gamma^*\}|. \end{aligned}$$

It follows that one of the following events is true for  $w$ :

- (a) At least  $n^{2^h-2}$  steps in  $w$  have a  $\Gamma_{\text{high}}$ -symbol on top of the stack. More formally,

$$|\{i \in \mathbb{N}_0 \mid w(i) \in \Gamma_{\text{high}}\Gamma^*\}| \geq n^{2^h-2}.$$

- (b) Event (a) is not true, but at least  $n^{2^{h+1}-2} - n^{2^h-2}$  steps in  $w$  have a  $\Gamma_{\text{low}}$ -symbol on top of the stack. More formally,

$$\begin{aligned} |\{i \in \mathbb{N}_0 \mid w(i) \in \Gamma_{\text{high}}\Gamma^*\}| &< n^{2^h-2} \quad \text{and} \\ |\{i \in \mathbb{N}_0 \mid w(i) \in \Gamma_{\text{low}}\Gamma^*\}| &\geq n^{2^{h+1}-2} - n^{2^h-2}. \end{aligned}$$

In order to give bounds on the probabilities of events (a) and (b), it is convenient to “reshuffle” the execution order of runs in the following way: Whenever a rule  $X \hookrightarrow \alpha$  is executed, we do not replace the  $X$ -symbol on top of the stack by  $\alpha$ , but instead we push only the  $\Gamma_{\text{high}}$ -symbols in  $\alpha$  on top of the stack, whereas the  $\Gamma_{\text{low}}$ -symbols in  $\alpha$  are added to the *bottom* of the stack. Since  $\Delta$  is a pBPA and thus does not have control states, the reshuffling of the execution order does not influence the distribution of the termination time. The advantage of this execution order is that each run can be decomposed into two phases:

- (1) In the first phase, the symbol on the top of the stack is always a  $\Gamma_{\text{high}}$ -symbol. When rules are executed,  $\Gamma_{\text{low}}$ -symbols may be produced, which are added to the bottom of the stack.

- (2) In the second phase, the stack consists of  $\Gamma_{low}$ -symbols exclusively. Notice that by definition of  $\Gamma_{low}$ , no new  $\Gamma_{high}$ -symbols can be produced.

In terms of those phases, the above events (a) and (b) can be reformulated as follows:

- (a) The first phase of  $w$  consists of at least  $n^{2^h-2}$  steps. The probability of this event is equal to

$$\mathcal{P}_{\Delta_{high}}(\mathbf{T}_{X_0} \geq n^{2^h-2}),$$

where  $\Delta_{high}$  is the pBPA obtained from  $\Delta$  by deleting all  $\Gamma_{low}$ -symbols from the right hand sides of the rules and deleting all rules with  $\Gamma_{low}$ -symbols on the left hand side, and  $\mathcal{P}_{\Delta_{high}}$  is its associated probability measure.

- (b) The first phase of  $w$  consists of fewer than  $n^{2^h-2}$  steps (which implies that at most  $n^{2^h-2}$   $\Gamma_{low}$ -symbols are produced during the first phase), and the second phase consists of at least  $n^{2^{h+1}-2} - n^{2^h-2}$  steps. Therefore, the probability of the event (b) is at most

$$\max \left\{ \mathcal{P}_{\Delta_{low}}(\mathbf{T}_{\alpha_0} \geq n^{2^{h+1}-2} - n^{2^h-2}) \mid \alpha_0 \in \Gamma_{low}^*, 1 \leq |\alpha_0| \leq n^{2^h-2} \right\},$$

where  $\Delta_{low}$  is the pBPA  $\Delta$  restricted to the  $\Gamma_{low}$ -symbols, and  $\mathcal{P}_{\Delta_{low}}$  is its associated probability measure. Notice that  $n^{2^{h+1}-2} - n^{2^h-2} \geq n^{2^{h+1}-2}/2$  for large enough  $n$ . Furthermore, by the definition of  $\Gamma_{low}$ , the SCCs of  $\Delta_{low}$  are all bottom SCCs. Hence, by Lemma 18, the above maximum is at most  $D/n$ .

Summing up, we have for almost all  $n \in \mathbb{N}$ :

$$\begin{aligned} \mathcal{P}(\mathbf{T}_{X_0} \geq n^{2^{h+1}-2}) &\leq \mathcal{P}(\text{event (a)}) + \mathcal{P}(\text{event (b)}) \\ &\leq \mathcal{P}_{\Delta_{high}}(\mathbf{T}_{X_0} \geq n^{2^h-2}) + D/n \quad (\text{as argued above}) \\ &\leq \frac{(h-1)D}{n} + \frac{D}{n} = \frac{hD}{n} \quad (\text{by the induction hypothesis}). \end{aligned}$$

This completes the induction proof.  $\square$

### 6.3 Proof of Proposition 11

The proof of Proposition 11 is similar to the proof of Proposition 10 from the previous subsection. Here is a restatement of Proposition 11.

**Proposition 11.** *Let  $\Delta$  be an almost surely terminating pBPA with stack alphabet  $\Gamma$ . Assume that  $X_0$  depends on all  $X \in \Gamma \setminus \{X_0\}$ . Assume  $E[X_0] = \infty$ . Then there is  $c > 0$  such that*

$$\frac{c}{\sqrt{n}} \leq \mathcal{P}(\mathbf{T}_{X_0} \geq n) \quad \text{for all } n \in \mathbb{N}.$$

*Proof.* For a square matrix  $M$  denote by  $\rho(M)$  the spectral radius of  $M$ , i.e., the greatest absolute value of its eigenvectors. Let  $A_\Delta$  be the matrix from the previous subsection. We claim:

$$\rho(A_\Delta) = 1. \tag{12}$$

The assumption that  $\Delta$  is almost surely terminating implies that  $\rho(A_\Delta) \leq 1$ , see, e.g., Section 8.1 of [20]. Assume for a contradiction that  $\rho(A_\Delta) < 1$ . Using standard theory of nonnegative matrices (see, e.g., [1]), this implies that the matrix inverse  $B := (I - A_\Delta)^{-1}$  (here,  $I$  denotes the identity matrix) exists; i.e.,  $B$  is finite in all components. It is shown in [16] that  $E[X_0] = (B \cdot \mathbf{1})(X_0)$  (here,  $\mathbf{1}$  denotes the vector with  $\mathbf{1}(X) = 1$  for all  $X$ ). This is a contradiction to our assumption that  $E[X_0] = \infty$ . Hence, (12) is proved.

It follows from (12) and standard theory of nonnegative matrices [1] that  $A_\Delta$  has a principal submatrix, say  $A'$ , which is irreducible and satisfies  $\rho(A') = 1$ . Let  $\Gamma'$  be the subset of  $\Gamma$  such that  $A'$  is obtained from  $A$  by deleting all rows and columns which are not indexed by  $\Gamma'$ . Let  $\Delta'$  be the pBPA with stack alphabet  $\Gamma'$  such that  $\Delta'$  is obtained from  $\Delta$  by removing all rules with symbols from  $\Gamma \setminus \Gamma'$  on the left hand side and removing all symbols from  $\Gamma \setminus \Gamma'$  from all right hand sides. Clearly,  $A_{\Delta'} = A'$ , so  $\rho(A_{\Delta'}) = 1$  and  $A_{\Delta'}$  is irreducible. Since  $\Delta'$  is a sub-pBPA of  $\Delta$  and  $X_0$  depends on all symbols in  $\Gamma'$ , it suffices to prove the proposition for  $\Delta'$  and an arbitrary start symbol  $X'_0 \in \Gamma'$ .

Therefore, w.l.o.g. we can assume in the following that  $A_\Delta = A$  is irreducible. Then it follows, using (12) and Perron-Frobenius theory [1], that there is a positive vector  $\mathbf{u} \in \mathbb{R}_+^\Gamma$  such that  $A \cdot \mathbf{u} = \mathbf{u}$ . W.l.o.g. we assume  $\mathbf{u}(X_0) = 1$ . Using Lemma 14 we can assume w.l.o.g. that  $\Delta$  is  $\mathbf{u}$ -progressive. (The pBPA  $\Delta$  may be relaxed.)

As in the proof of Proposition 15, for each  $X \in \Gamma$  we define a function  $g_X : \mathbb{R} \rightarrow \mathbb{R}$  by setting

$$g_X(\theta) := \sum_{X \xrightarrow{p} \alpha} p \cdot \exp(-\theta \cdot (-\mathbf{u}(X) + \#(\alpha) \cdot \mathbf{u})).$$

The following lemma states some properties of  $g_X$ .

**Lemma 19.** *The following holds for all  $X \in \Gamma$ :*

- (a) For all  $\theta > 0$  we have  $1 = g_X(0) < g_X(\theta)$ .
- (b) For all  $\theta > 0$  we have  $0 = g'_X(0) < g'_X(\theta)$ .
- (c) For all  $\theta \geq 0$  we have  $0 < g''_X(\theta)$ .
- (d) There is  $c_2 > 0$  such that for all  $0 < \theta \leq 1$  we have  $g'_X(\theta) \leq c_2 \theta$ .
- (e) There is  $c_3 > 1$  such that for all  $n \in \mathbb{N}$  we have  $g_X(1/\sqrt{n})^n \geq c_3$ .
- (f) There is  $c_4 > 0$  such that for all  $n \in \mathbb{N}$  we have  $\frac{1/n}{1 - 1/g_X(1/\sqrt{n})} \leq c_4$ .

*Proof (of the lemma).* The proof of items (a)–(c) follows exactly the proof of Lemma 16 and is therefore omitted. (For the equality  $0 = g'_X(0)$  in (b) one uses  $A \cdot \mathbf{u} = \mathbf{u}$ .)

- (d) It suffices to prove that  $g'_X(\theta)/\theta$  is bounded for  $\theta \rightarrow 0$ . Using l'Hopital's rule we have  $\lim_{\theta \rightarrow 0} g'_X(\theta)/\theta = g''_X(0) > 0$ .
- (e) Clearly, we have  $g_X(1/\sqrt{n})^n > 1$  for all  $n$ . Furthermore, we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \ln g_X(1/\sqrt{n})^n &= \lim_{n \rightarrow \infty} \frac{\ln g_X(n^{-1/2})}{1/n} \\ &= \frac{1}{2} \lim_{n \rightarrow \infty} \frac{g'_X(n^{-1/2})}{n^{-1/2}} && \text{(l'Hopital's rule)} \\ &= \frac{g''_X(0)}{2} && \text{(l'Hopital's rule)} \\ &> 0 && \text{(by (c))} \end{aligned}$$

Hence the claim follows.

(f) The claim follows again from l'Hopital's rule:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1/n}{1 - 1/g_X(n^{-1/2})} &= \lim_{n \rightarrow \infty} \frac{-1/n^2}{(1/g_X(n^{-1/2}))^2 \cdot g'_X(n^{-1/2}) \cdot (-1/2)n^{-3/2}} \\ &= \lim_{n \rightarrow \infty} \frac{2n^{-1/2}}{g'_X(n^{-1/2})} = \frac{2}{g''_X(0)} < \infty \end{aligned}$$

This completes the proof of the lemma.  $\square$

Let in the following  $\theta > 0$ . As in the proof of Proposition 15, given a run  $w \in \text{Run}(X_0)$  and  $i \geq 0$ , we write  $X^{(i)}(w)$  for the symbol  $X \in \Gamma$  for which  $w(i) = X\alpha$ . Define

$$m_\theta^{(i)}(w) = \begin{cases} \exp(-\theta \cdot \#(w(i)) \cdot \mathbf{u}) \cdot \prod_{j=0}^{i-1} \frac{1}{g_{X^{(j)}(w)}(\theta)} & \text{if } i = 0 \text{ or } w(i-1) \neq \varepsilon \\ m_\theta^{(i-1)}(w) & \text{otherwise} \end{cases}$$

As in Lemma 17, one can show that the sequence  $m_\theta^{(0)}, m_\theta^{(1)}, \dots$  is a martingale. As in the proof of Proposition 15, Doob's Optional-Stopping Theorem implies  $\exp(-\theta) = m_\theta^{(0)} = \mathbb{E} \left[ m_\theta^{(\mathbf{T}_{X_0})} \right]$ . Hence we have for each  $n \in \mathbb{N}$  (writing  $\mathbf{T}$  for  $\mathbf{T}_{X_0}$ ):

$$\begin{aligned} \exp(-\theta) &= \mathbb{E} \left[ m_\theta^{(\mathbf{T})} \right] && \text{(by optional-stopping)} \\ &= \mathbb{E} \left[ \exp(-\theta \cdot 0) \cdot \prod_{j=0}^{\mathbf{T}-1} \frac{1}{g_{X^{(j)}}(\theta)} \right] \\ &= \mathbb{E} \left[ \prod_{j=0}^{\mathbf{T}-1} \frac{1}{g_{X^{(j)}}(\theta)} \right] \end{aligned}$$

Taking, on both sides, the derivative with respect to  $\theta$  yields

$$\exp(-\theta) \leq \sum_{i=1}^{\infty} i \cdot \mathcal{P}(\mathbf{T} = i) \cdot \frac{g'_{1,\theta}(\theta)}{g_{0,\theta}(\theta)^{i+1}}, \quad (13)$$

where  $g_{0,\theta} = g_X$  and  $g_{1,\theta} = g_Y$  for some  $X, Y \in \Gamma$  possibly depending on  $\theta$ . The following lemma bounds an “upper” subseries of the right-hand-side of (13).

**Lemma 20.** *For all  $\varepsilon > 0$  there is a  $a \in \mathbb{N}$  such that for all  $n \in \mathbb{N}$  and  $\theta = 1/\sqrt{n}$  we have*

$$\sum_{i=an+1}^{\infty} i \cdot \mathcal{P}(\mathbf{T} = i) \cdot \frac{g'_{1,\theta}(\theta)}{g_{0,\theta}(\theta)^{i+1}} \leq \varepsilon.$$

*Proof (of the lemma).* By rearranging the series we get for all  $n \in \mathbb{N}$  and  $\theta = 1/\sqrt{n}$ :

$$\begin{aligned}
& \sum_{i=an+1}^{\infty} i \cdot \mathcal{P}(\mathbf{T} = i) \cdot \frac{g'_{1,\theta}(\theta)}{g_{0,\theta}(\theta)^{i+1}} \\
& \leq \sum_{i=0}^{an-1} \frac{\mathcal{P}(\mathbf{T} > an) \cdot g'_{1,\theta}(\theta)}{g_{0,\theta}(\theta)^{an+2}} + \sum_{i=an}^{\infty} \frac{\mathcal{P}(\mathbf{T} > i) \cdot g'_{1,\theta}(\theta)}{g_{0,\theta}(\theta)^{i+2}} \\
& \leq \underbrace{\frac{an \cdot \mathcal{P}(\mathbf{T} > an) \cdot g'_{1,\theta}(\theta)}{g_{0,\theta}(\theta)^{an}}}_{=:q_1} + \underbrace{\sum_{i=an}^{\infty} \frac{\mathcal{P}(\mathbf{T} > i) \cdot g'_{1,\theta}(\theta)}{g_{0,\theta}(\theta)^i}}_{=:q_2}
\end{aligned}$$

We bound  $q_1$  and  $q_2$  separately. By Proposition 10 there is  $c_1 > 0$  such that  $\mathcal{P}(\mathbf{T} > k) \leq c_1/\sqrt{k}$ . Hence we have, using Lemma 19 (d), (e):

$$\begin{aligned}
q_1 & \leq \frac{\sqrt{an} \cdot c_1 \cdot c_2 / \sqrt{n}}{c_3^a} \leq \frac{c_1 c_2 \sqrt{a}}{c_3^a}, & \text{and similarly,} \\
q_2 & \leq \frac{c_1}{\sqrt{an}} \cdot \frac{c_2}{\sqrt{n}} \cdot \sum_{i=an}^{\infty} \frac{1}{g_{0,\theta}(\theta)^i} \\
& = \frac{c_1 c_2}{\sqrt{a} \cdot n \cdot g_{0,\theta}(\theta)^{an} \cdot (1 - 1/g_{0,\theta}(\theta))} \\
& \leq \frac{c_1 c_2 c_4}{\sqrt{a} \cdot c_3^a} & (\text{by Lemma 19 (e), (f)}).
\end{aligned}$$

These bounds on  $q_1$  and  $q_2$  can be made arbitrarily small by choosing  $a$  large enough. This completes the proof of the lemma.  $\square$

This lemma implies a first lower bound on the distribution of  $\mathbf{T}$ :

**Lemma 21.** *For any  $c > 0$  there is  $s \in \mathbb{N}$  such that for all  $n \in \mathbb{N}$  we have:*

$$\sum_{i=1}^{sn} i \cdot \mathcal{P}(\mathbf{T} = i) \geq c\sqrt{n}.$$

*Proof (of the lemma).* Let  $a \in \mathbb{N}$  be the number from Lemma 20 for  $\varepsilon = \exp(-1)/2$ . For all  $n \in \mathbb{N}$  and  $\theta = 1/\sqrt{n}$  we have:

$$\begin{aligned}
& g'_{1,\theta}(\theta) \cdot \sum_{i=1}^{an} i \cdot \mathcal{P}(\mathbf{T} = i) \\
& \geq \sum_{i=1}^{an} i \cdot \mathcal{P}(\mathbf{T} = i) \cdot \frac{g'_{1,\theta}(\theta)}{g_{0,\theta}(\theta)^{i+1}} \\
& \geq \exp(-\theta) - \varepsilon & (\text{by (13) and Lemma 20}) \\
& \geq \exp(-1) - \varepsilon = \varepsilon & (\text{by the choice of } \varepsilon),
\end{aligned}$$

so, with Lemma 19 (d) we have for all  $n \in \mathbb{N}$ :

$$\sum_{i=1}^{an} i \cdot \mathcal{P}(\mathbf{T} = i) \geq \frac{\varepsilon}{c_2} \sqrt{n}.$$

For the given number  $c > 0$ , choose  $s := a \lceil cc_2/\varepsilon \rceil^2$ . Then it follows for all  $m \in \mathbb{N}$ :

$$\sum_{i=1}^{sm} i \cdot \mathcal{P}(\mathbf{T} = i) \geq c\sqrt{m},$$

which proves the lemma.  $\square$

Now we can complete the proof of the proposition. By Proposition 10 there is  $c_1 > 0$  such that  $\mathcal{P}(\mathbf{T} > n) \leq c_1/\sqrt{n}$  for all  $n \in \mathbb{N}$ . By Lemma 21, there is  $s \in \mathbb{N}$  such that

$$\sum_{i=1}^{sn} i \cdot \mathcal{P}(\mathbf{T} = i) \geq (2c_1 + 2)\sqrt{n} \quad \text{for all } n \in \mathbb{N}.$$

We have for all  $n \in \mathbb{N}$ :

$$\begin{aligned} \sum_{i=n}^{sn} i \cdot \mathcal{P}(\mathbf{T} = i) &\geq \sum_{i=1}^{sn} i \cdot \mathcal{P}(\mathbf{T} = i) - \sum_{i=1}^n i \cdot \mathcal{P}(\mathbf{T} = i) \\ &\geq (2c_1 + 2)\sqrt{n} - \sum_{i=0}^n \mathcal{P}(\mathbf{T} > i) && \text{(by the choice of } s \text{ above)} \\ &\geq (2c_1 + 2)\sqrt{n} - 1 - \sum_{i=1}^n \frac{c_1}{\sqrt{i}} && \text{(by the choice of } c_1 \text{ above)} \\ &\geq (2c_1 + 1)\sqrt{n} - \int_0^n \frac{c_1}{\sqrt{i}} di \\ &= (2c_1 + 1)\sqrt{n} - 2c_1\sqrt{n} \\ &= \sqrt{n} \end{aligned}$$

It follows:

$$\begin{aligned} sn\mathcal{P}(\mathbf{T} \geq n) &\geq sn \sum_{i=n}^{sn} \mathcal{P}(\mathbf{T} = i) \geq \sum_{i=n}^{sn} i \cdot \mathcal{P}(\mathbf{T} = i) \\ &\geq \sqrt{n} && \text{(by the computation above)} \end{aligned}$$

Hence we have

$$\mathcal{P}(\mathbf{T} \geq n) \geq \frac{1/s}{\sqrt{n}},$$

which completes the proof of the proposition.  $\square$

## 6.4 Proof of Proposition 12

Here is a restatement of Proposition 12.

**Proposition 12.** *Let  $\Delta_h$  be the pBPA with  $\Gamma_h = \{X_1, \dots, X_h\}$  and the following rules:*

$$X_h \xrightarrow{1/2} X_h X_h, X_h \xrightarrow{1/2} X_{h-1}, \dots, X_2 \xrightarrow{1/2} X_2 X_2, X_2 \xrightarrow{1/2} X_1, X_1 \xrightarrow{1/2} X_1 X_1, X_1 \xrightarrow{1/2} \varepsilon$$

*Then  $[X_h] = 1$ ,  $E[X_h] = \infty$ , and there is  $c_h > 0$  with*

$$\frac{c_h}{n^{1/2^h}} \leq \mathcal{P}(\mathbf{T}_{X_h} \geq n) \quad \text{for all } n \in \mathbb{N}.$$

*Proof.* Observe that the third statement implies the second statement, since

$$E[X_h] = \sum_{n=1}^{\infty} \mathcal{P}(\mathbf{T}_{X_h} \geq n) \geq \sum_{n=1}^{\infty} c_h \cdot n^{-1/2^h} \geq \sum_{n=1}^{\infty} c_h/n = \infty.$$

We proceed by induction on  $h$ . Let  $h = 1$ . The pBPA  $\Delta_1$  is equivalent to a random walk on  $\{0, 1, 2, \dots\}$ , started at 1, with an absorbing barrier at 0. It is well-known (see, e.g., [11]) that the probability that the random walk finally reaches 0 is 1, but that there is  $c_1 > 0$  such that the probability that the random walk has not reached 0 after  $n$  steps is at least  $c_1/\sqrt{n}$ . Hence  $[X_1] = 1$  and  $\mathcal{P}(\mathbf{T}_{X_1} \geq n) \geq c_1/\sqrt{n} = c_1 \cdot n^{-1/2}$ .

Let  $h > 1$ . The behavior of  $\Delta_h$  can be described in terms of a random walk  $W_h$  whose states correspond to the number of  $X_h$ -symbols in the stack. Whenever an  $X_h$ -symbol is on top of the stack, the total number of  $X_h$ -symbols in the stack increases by 1 with probability 1/2, or decreases by 1 with probability 1/2, very much like the random walk equivalent to  $\Delta_1$ . In the second case (i.e., the rule  $X_h \hookrightarrow X_{h-1}$  is taken), the random walk  $W_h$  resumes only after a run of  $\Delta_{h-1}$  (started with a single  $X_{h-1}$ -symbol) has terminated. By the induction hypothesis,  $[X_{h-1}] = 1$ , so with probability 1 all spawned “sub-runs” of  $\Delta_{h-1}$  terminate. Since  $W_h$  also terminates with probability 1, it follows  $[X_h] = 1$ .

It remains to show that there is  $c_h > 0$  with  $\mathcal{P}(\mathbf{T}_{X_h} \geq n) \geq c_h \cdot n^{-1/2^h}$  for all  $n \geq 1$ . Consider, for any  $n \geq 1$  and any  $\ell > 0$ , the event  $A_\ell$  that  $W_h$  needs at least  $\ell$  steps to terminate (not counting the steps of the spawned sub-runs) and that at least one of the spawned sub-runs needs at least  $n$  steps to terminate. Clearly,  $\mathbf{T}_{X_h}(w) \geq n$  holds for all  $w \in A_\ell$ , so it suffices to find  $c_h > 0$  so that for all  $n \geq 1$  there is  $\ell > 0$  with  $\mathcal{P}(A_\ell) \geq c_h \cdot n^{-1/2^h}$ . At least half of the steps of  $W_h$  are steps down, so whenever  $W_h$  needs at least  $2\ell$  steps to terminate, it spawns at least  $\ell$  sub-runs. It follows:

$$\begin{aligned} \mathcal{P}(A_\ell) &\geq \mathcal{P}(W_h \text{ needs at least } 2\ell \text{ steps}) \cdot \left(1 - (\mathcal{P}(\mathbf{T}_{X_{h-1}} < n))^\ell\right) \\ &\geq \frac{c_1}{\sqrt{2\ell}} \cdot \left(1 - \left(1 - c_{h-1} \cdot n^{-1/2^{h-1}}\right)^\ell\right) \quad (\text{by induction hypothesis}) \end{aligned}$$

Now we fix  $\ell := n^{1/2^{h-1}}$ . Then the second factor of the product above converges to  $1 - e^{-c_{h-1}}$  for  $n \rightarrow \infty$ , so for large enough  $n$

$$\mathcal{P}(A_\ell) \geq \frac{c_1}{2} \cdot (1 - e^{-c_{h-1}}) \cdot n^{-1/2^h}.$$



Hence, we can choose  $c_h < \frac{c_1}{2} \cdot (1 - e^{-c_h-1})$  such that  $\mathcal{P}(A_\ell) \geq c_h \cdot n^{-1/2^h}$  holds for all  $n \geq 1$ .  $\square$

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